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Best Approximation by Periodic Smooth Functions*

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Let $\tilde{W}_n$ be the set of 2π-periodic functions with absolutely continuous $(n-1)$th derivatives and nth derivatives with essential suprema bounded by one. Let $n>1$. Best uniform approximations to a periodic continuous function from $\tilde{W}_n$ are characterized. The result depends upon an analysis of the relation between the zeros, knots, and signs of periodic splines with simple knots. An appendix by O. V. Davydov states an alternative characterisation and demonstrates that the two characterisations are equivalent.

1. INTRODUCTION

Throughout this paper "periodic" will mean periodic with period 2π. Let $\tilde{C}$ be the space of periodic continuous real valued functions defined on the real line $\mathbb{R}$, and let $\tilde{C}$ be equipped with the uniform norm. For each $n \in \mathbb{N}$ let $\tilde{W}_n$ be the set of those functions $u \in \tilde{C}$ which have absolutely continuous $(n-1)$th derivative $u^{(n-1)}$, and whose nth derivatives satisfy the condition $\|u^{(n)}\|_\infty \leqslant 1$. The paper presents a characterization (Theorem 5.2.1), when $n>1$, of those $u \in \tilde{W}_n$ which are best approximations from $\tilde{W}_n$ to a given $v_0 \in \tilde{C}$.

Let $C([0,1])$ be the space of continuous real valued functions defined on the interval $[0,1]$, equipped with the uniform norm, and let W_n be the set of those functions $u \in C([0,1])$ which have absolutely continuous $(n-1)$th derivatives, and whose nth derivatives satisfy the condition $\|u^{(n)}\|_\infty \leqslant 1$. Korneichuk in 1961 [11] gave a characterization of best approximations $u \in W_1$ to a given to $v_0 \in C([0,1])$. Sattes [13] gave a characterization for the cases $n>1$.

* The paper presents the central results of the author's thesis [12]. The author has been prevented by illness from further work on his draft of the paper and it has been revised by A. L. Brown, who was the author's research supervisor. An appendix to the paper has been contributed by O. V. Davydov. Correspondence and reprint requests should be sent care of A. L. Brown, D11 Type IV, IMTECH Colony, Sector 39A, Chandigarh, 160036, India, who will act on behalf of and in consultation with the author.

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Let $n > 1$. Sattes showed that for each $v_0 \in C([0, 1]) \backslash W_n$, there exists a subinterval $[a, b]$ of $[0, 1]$ such that best approximations $u_0 \in W_n$ to v_0 are characterized in terms of $\|v_0 - u_0\|$ and a precisely defined behaviour of u_0 and $v_0 - u_0$ on the interval $[a, b]$. It follows from the characterization that a best approximation is, on the interval $[a, b]$, a uniquely determined perfect spline. This "interval of uniqueness" of best approximations may be a proper subinterval of $[0, 1]$. It also follows from Sattes's result that his characterisation also provides a sufficient condition for the periodic problem, that is, Sattes's condition is sufficient to ensure that $u_0 \in \tilde{W}_n$ is a best approximation from $\tilde{W}_n$ to a $v_0 \in \tilde{C} \backslash \tilde{W}_n$. The condition, however, is not necessary. In the periodic problem there are two cases. The definitions of the two cases are technical and are given in Section 4.1. If $v_0 \in \tilde{C} \backslash \tilde{W}_n$ belongs to Case I then there exists a subinterval $[a, b]$ of $[0, 2\pi)$ on which Sattes's non-periodic condition is satisfied. However, if $v_0 \in \tilde{C} \backslash \tilde{W}_n$ belongs to Case II then the situation is essentially periodic, there is a unique best approximation to v_0 from $\tilde{W}_n$ which is a periodic perfect spline, and the characterization of the best approximation involves conditions which do not appear in the non-periodic problem. It is shown in [12] that Sattes's theorem can be deduced from the periodic Theorem 5.2.1; that is, the periodic problem contains but is not contained in the non-periodic problem. It is not difficult to construct examples to show that none of the conditions of the characterization theorem (Case I, and two subcases of Case II) are redundant. The case $n = 1$ is not considered; it is relatively very simple and the periodic and non-periodic problems are equivalent.

An alternative approach to Sattes's theorem was given by Brown [3]. Functions in the set W_n have integral representations, given by Taylor's theorem with integral remainder, involving the truncated power kernel

$$K_n(x - y) = \frac{(x - y)_+^{n-1}}{(n-1)!}.$$

A duality argument first given in a simple form by Glashoff [10] yields a general and preliminary characterization theorem (Theorem 1 of [3]) for best approximations from certain sets defined by integral operators. Sattes's theorem for best approximations from W_n was deduced from the general theorem. The deduction involved an analysis of the zeros and signs of functions (called here *associated functions*) which are of the form

$$w(y) = \int_0^1 K_n(x - y) \, d\lambda(x),$$

where λ is a measure which annihilates polynomials of degree $\leqslant n - 1$. If the measure λ has finite support then the function $w(y)$ is a spline function of degree $\leqslant n-1$ with simple knots and compact support. The analysis then also involves the knots of the splines.

A function $u \in \tilde{W}_n$ has an integral representation

$$u = c + D_n * u^{(n)},$$

where D_n is a well-known convolution kernel and $c \in \mathbb{R}$ is the mean value of u. The function D_n is a periodic spline of degree n with knots at the points of $2\pi\mathbb{Z}$ (that is, it corresponds to a spline function on the circle which has a single knot); it is referred to in [9] as a *Bernoulli spline*. The functions D_n, for $n \geqslant 1$, are discussed in Section 2.1. Through the integral representation (translated to the circle) the general characterization theorem of [3] is applicable and its specialization to approximation in $\tilde{C}$ from $\tilde{W}_n$ is stated as Theorem 3.1.1. Separating measures and associated functions for a function $v_0 \in \tilde{C} \backslash \tilde{W}_n$ are then defined. It is shown (Theorem 3.1.7) that for each $v_0 \in \tilde{C} \backslash \tilde{W}_n$ there exist separating measures of finite support for which the associated functions are periodic splines of degree $\leqslant n-1$. Case I, referred to above, is the case in which there exists a separating measure of finite support such that there is an associated function with a zero interval. Case II is "not Case I."

Section 4 is devoted to an analysis of the knots, zeros, and signs of periodic splines of degree $\leqslant n-1$ with simple knots. Section 4.1 is concerned with splines with a zero interval, and Section 4.2 with periodic splines which have no zero interval. The analysis is limited to results which are necessary for the characterization Theorem 5.2.1.

Section 2 presents preliminary material: the convolution kernels D_n, the integral representations, and a result giving bounds on the number of zeros of splines (Theorem 2.2.2) which is used repeatedly in later sections.

2. PRELIMINARY RESULTS

2.1. *The Convolution Kernels D_n*

First we require some definitions. Let $\tilde{\mathscr{M}}$ denote the space of real valued regular Borel measures λ on $\mathbb{R}$ which are periodic, that is, such that $\lambda(B + 2\pi) = \lambda(B)$ for all Borel measurable $B \subseteq \mathbb{R}$, and such that $|\lambda|([0, 2\pi)) < \infty$. Let $\tilde{\mathscr{M}}_0$ be the set of $\lambda \in \tilde{\mathscr{M}}$ such that $\lambda([0, 2\pi)) = 0$. All the measures which appear in the discussion will belong to $\tilde{\mathscr{M}}_0$ either by construction or by assumption.

The space $\tilde{\mathscr{M}}$, normed by

$$\|\lambda\| = |\lambda|([0, 2\pi)),$$

can be identified with the dual of the space $\tilde{C}$.

A subset $\tilde{K}$ of $\mathbb{R}$ will be called a *periodic set* if $\tilde{K}+2\pi=\tilde{K}$. If $\tilde{K}$ is a periodic set which corresponds to a finite subset of the circle consisting of m points then $\tilde{K}$ can be written as $\tilde{K}=\{x_j: j\in\mathbb{Z}\}$ where

$$x_j < x_{j+1} \qquad \text{and} \qquad x_{j+m}=x_j+2\pi \qquad \text{for all} \quad j\in\mathbb{Z}.$$

If $\lambda\in\tilde{\mathscr{M}}$ then supp λ is a periodic set.

If D is a bounded periodic Borel measurable function and $\lambda\in\tilde{\mathscr{M}}$ then the convolution $D*\lambda$ is defined by

$$(D*\lambda)(x)=\int_{[0,\,2\pi)} D(x-y)\,d\lambda(y).$$

If λ is a measure defined by $d\lambda(y)=f(y)\,dy$, where f is periodic and integrable on $[0,2\pi]$, then $D*\lambda$ is just the convolution $D*f$ of the two functions D and f.

If f, g, h are periodic functions which are Lebesgue integrable on $[0,2\pi]$ then $f*(g*h)=(f*g)*h$. If f, g are bounded periodic Borel measurable functions and $\lambda\in\tilde{\mathscr{M}}$ then $(f*g)*\lambda=f*(g*\lambda)$.

Now define D_1 to be the periodic function such that

$$D_1(x)=\frac{\pi-x}{2\pi}$$

for $0\leqslant x<2\pi$. Define D_n for $n\geqslant 2$ to be the convolution powers of D_1, that is,

$$D_n=D_{n-1}*D_1.$$

The importance of these functions lies in the fact that convolution with D_1 is an integration operator. The first theorem gives a more general result.

2.1.1. THEOREM.

(i) *If $\lambda\in\tilde{\mathscr{M}}_0$ then, for all $x<x'$,*

$$(D_1*\lambda)(x')-(D_1*\lambda)(x)=\lambda((x,x']).$$

(ii) *If f is periodic, is integrable on $[0,2\pi]$, and has mean value zero then, for all $x<x'$,*

$$(D_1*f)(x')-(D_1*f)(x)=\int_x^{x'} f(y)\,dy.$$

Consequently, D_1*f is that periodic integral of f which has mean value zero.

Proof. A convolution is given by integration over any half-open interval of length 2π, so, for all $x \in \mathbb{R}$,

$$(D_1 * \lambda)(x) = \int_{(x-2\pi,\, x]} D_1(x-y)\, d\lambda(y)$$
$$= \int_{(x-2\pi,\, x]} \frac{\pi-(x-y)}{2\pi}\, d\lambda(y)$$
$$= \int_{(x-2\pi,\, x]} \frac{y}{2\pi}\, d\lambda(y).$$

Therefore, if $x < x'$ then

$$(D_1 * \lambda)(x') - (D_1 * \lambda)(x) = \int_{(x,\, x']} \frac{y}{2\pi}\, d\lambda(y) - \int_{(x-2\pi,\, x'-2\pi]} \frac{y}{2\pi}\, d\lambda(y)$$
$$= \int_{(x,\, x']} \frac{y}{2\pi}\, d\lambda(y) - \int_{(x,\, x']} \frac{y-2\pi}{2\pi}\, d\lambda(y)$$
$$= \lambda((x, x']).$$

This proves (i). If λ is the measure defined by $d\lambda(y) = f(y)\, dy$ then (i) gives (ii).

Simple and well-known properties of the functions D_n are listed in the next proposition.

2.1.2. PROPOSITION. (i) *For each $n \in \mathbb{N}$ the function D_n has mean value zero, that is,*

$$\frac{1}{2\pi} \int_0^{2\pi} D_n(t)\, dt = 0.$$

(ii) $D_1(t) + D_1(-t) = \chi_{2\pi\mathbb{Z}}(t)$ *for all* $t \in \mathbb{R}$,

$$D_n(-t) = (-1)^n D_n(t) \qquad \textit{for all} \quad t \in \mathbb{R} \quad \textit{and all} \quad n > 1.$$

(iii) *D_n is a periodic spline of degree n with knots at the points of $2\pi\mathbb{Z}$ and, if $n \geqslant 2$, has a continuous $(n-2)$th derivative.*

Next we consider integral representations of functions in $\tilde{W}_n$ and of periodic splines with simple knots.

2.1.3 THEOREM. *If $n \in \mathbb{N}$ and $u \in \tilde{C}$ then $u \in \tilde{W}_n$ if and only if*

$$u = c + D_n * f$$

for some $c \in \mathbb{R}$ *and some essentially bounded measurable periodic function* f *of mean value zero such that* $\|f\|_\infty \leqslant 1$.

Proof. $D_n * f = D_1(D_1(\ldots(D_1 * f)\ldots))$ is, by Theorem 2.1.1, an nth integral of f, and so, if $\|f\|_\infty \leqslant 1$, is an element of $\widetilde{W}_n$. If $u \in \widetilde{W}_n$ and $f = u^{(n)}$ then $u - D_n * f$ has an nth derivative which almost everywhere exists and is zero, so that $u - D_n * f$ is a periodic polynomial which must be constant.

2.1.4. *Notation.* If $(g, \lambda) \in \mathbb{R} \times \widetilde{\mathscr{M}}_0$ then let

$$w_n(g, \lambda) = g + (-1)^n D_n * \lambda.$$

The factor $(-1)^n$ enters naturally in the context of Section 3 and it is convenient to introduce it at this point. Functions of this form are ever-present in the subsequent discussion.

The following simple results will be required.

2.1.5. Proposition. *Let* $n \in \mathbb{N}$. *Suppose* $(g, \lambda) \in \mathbb{R} \times \widetilde{\mathscr{M}}_0$.

(i) *If* $a < b$ *then* $\operatorname{supp} \lambda \cap (a, b) = \varnothing$ *if and only if the restriction of* $w_n(g, \lambda)$ *to* (a, b) *is a polynomial of degree* $\leqslant n-1$.

(ii) $w_n(g, \lambda) = 0$ *if and only if* $g = 0$ *and* $\lambda = 0$.

(iii) *If* $\operatorname{supp} \lambda^+ \cap \operatorname{supp} \lambda^- = \varnothing$ *then* $w_n(g, \lambda)$ *is a piecewise monotonic function.*

Proof.

(i) The $(n-1)$th derivative of $w_n(g, \lambda)$ is, by Theorem 2.1.1, given by

$$w_n(g, \lambda)^{(n-1)}(y) = (-1)^n (D_1 * \lambda)(y) = (-1)^n ((D_1 * \lambda)(a) + \lambda((a, y]))$$

for all $y > a$, and is constant on (a, b) if and only if $\operatorname{supp} \lambda \cap (a, b) = \varnothing$. This proves (i).

(ii) follows from (i).

(iii) If $\lambda \geqslant 0$ or $\lambda \leqslant 0$ then, by Theorem 2.1.1, $D_1 * \lambda$ is monotonic. Suppose $\operatorname{supp} \lambda^+ \cap \operatorname{supp} \lambda^- = \varnothing$. If $a \notin \operatorname{supp} \lambda$ then the interval $[a, a+2\pi]$ can be divided by points $a = x_0 < x_1 < \cdots < x_m = a + 2\pi$ so that $\operatorname{supp} \lambda^+ \cap [a, a+2\pi]$ and $\operatorname{supp} \lambda^- \cap [a, a+2\pi]$ are each contained in one of $(x_0, x_1) \cup (x_2, x_3) \cup \cdots$ and $(x_1, x_2) \cup (x_3, x_4) \cup \cdots$. It follows, by Theorem 2.1.1, that $D_1 * \lambda$ is piecewise monotonic. It then follows easily that, in all cases, $D_n * \lambda$, which is a repeated integral of $D_1 * \lambda$, is also piecewise monotonic.

2.1.6. THEOREM. *Let* $n>1$. *A function* $w \in \tilde{C}$ *is a periodic spline of degree* $n-1$ *with simple knots if and only if* $w = w_n(g, \lambda)$ *for some* $(g, \lambda) \in \mathbb{R} \times \tilde{\mathcal{M}}_0$ *such that* supp $\lambda \in \cap [0, 2\pi)$ *is finite.*

Proof. Suppose that $(g, \lambda) \in \mathbb{R} \times \tilde{\mathcal{M}}_0$ and that supp $\lambda \cap [0, 2\pi)$ is finite. Then $\tilde{K} = \text{supp}\, \lambda$ can be written as $\tilde{K} = \{x_j : j \in \mathbb{Z}\}$ where

$$x_j < x_{j+1} \qquad \text{and} \qquad x_{j+m} = x_j + 2\pi \qquad \text{for all} \quad j \in \mathbb{Z}.$$

In this case

$$w_n(g, \lambda)(y) = g + \sum_{j=1}^{m} (-1)^n D_n(y - x_j)\, \lambda(x_j)$$

and

$$w_n(g, \lambda)^{(n-1)}(y) = \sum_{j=1}^{m} (-1)^n D_1(y - x_j)\, \lambda(x_j)$$

for each $y \notin \{x_j : j \in \mathbb{Z}\}$. It follows from the fact that $\lambda \in \tilde{\mathcal{M}}_0$ that $w_n(g, \lambda)^{(n)}(y)$ exists and is zero for each $y \in \mathbb{R} \backslash \tilde{K}$. The $(n-2)$th derivative of $w_n(g, \lambda)$ is continuous. Thus $w_n(g, \lambda)$ is a spline of degree $n-1$ with simple knots at the points of supp λ. Furthermore,

$$w_n(g, \lambda)_+^{(n-1)}(x_j) - w_n(g, \lambda)_-^{(n-1)}(x_j) = (-1)^n \lambda(x_j) \tag{2.1.1}$$

for each $j \in \mathbb{Z}$.

Conversely, suppose that w is a periodic spline of degree $n-1$ with simple knots at the points of a set $\tilde{K} = \{x_j : j \in \mathbb{Z}\}$ as above. Let $\lambda \in \tilde{\mathcal{M}}$ with supp $\lambda = \tilde{K}$ be defined by (2.1.1). Then necessarily $\lambda \in \tilde{\mathcal{M}}_0$ and $w - (-1)^n D_n * \lambda$ has a continuous $(n-1)$th derivative, so is a periodic polynomial and is therefore constant. The proof is complete.

2.2. *On the Zeros of Periodic Splines*

This section will depend upon the definitions and results of [4]. In that paper the classes of functions considered are larger than the classes of splines which they contain. However, it is enough here to restrict attention to spline functions. Let $\mathcal{S}_{n-1}$ be the set of spline functions defined on $\mathbb{R}$ which are of degree $n-1$ and have a finite set of simple knots. Let $\tilde{\mathcal{S}}_{n-1}$ be the set of periodic spline functions defined on $\mathbb{R}$ which are of degree $n-1$ and have simple knots, finitely many in $[0, 2\pi)$. Throughout this section attention is restricted, in the interests of brevity, to the cases $n>1$. If $w \in \mathcal{S}_{n-1}$ or $w \in \tilde{\mathcal{S}}_{n-1}$ then $w^{(n-2)}$ is continuous and piecewise monotonic.

If w is a continuous function then it will be said that $[a, b]$ is a *zero interval* of w if $a<b$ and $w(y)=0$ for all $y \in [a, b]$. If $w \in \mathcal{S}_{n-1}$ or $w \in \tilde{\mathcal{S}}_{n-1}$

and $w(y)=0$ let $Z_n(w, y)$ be the multiplicity of the zero y of w as defined in [4]. That is, if $1 \leqslant \alpha \leqslant n-2$ and

$$w(y)=w^{(1)}(y)= \cdots =w^{(\alpha-1)}(y)=0, \qquad w^{(\alpha)}(y) \neq 0$$

then $Z_n(w, y)=\alpha$. If

$$w(y)=w^{(1)}(y)= \cdots =w^{(n-2)}(y)=0$$

then $Z_n(w, y)$ is either $n-1$ if w changes sign at y or n if w does not change sign at y. It follows that if y is a point of a zero interval of w then $Z_n(w, y)=n$.

Distinct zeros y and y' of w are said to be *separated zeros* of w if the interval with end points y and y' is not a zero interval of w. If I is an interval of $\mathbb{R}$ then $Z_n(w, I)$ will denote the maximal number of separated zeros of w on I, each zero being counted according to its multiplicity. If $w \in \tilde{\mathscr{S}}_{n-1}$ and $w \neq 0$ let $\tilde{Z}_n(w)=Z_n(w, a, a+2\pi))$ where a is any point which is not the right-hand endpoint of a zero interval of w (clearly the definition is unambiguous).

The following proposition ([4, Corollary 3.2]) is required.

2.2.1. Proposition. *Let $n>1$. If $w \in \mathscr{S}_{n-1}$ or $w \in \tilde{\mathscr{S}}_{n-1}$, $w(y_0)=0$ and $\alpha=Z_n(w, y_0) \leqslant n-1$ then $(y-y_0)^{\alpha}\, w(y)$ does not change sign at y_0. In particular, if $Z_n(w, y)=1$ then w changes sign at y_0.*

The next target is a theorem which relates the zero counts of a spline $w_n(g, \lambda)$ to the sign changes of the measure λ. If $\alpha_1, ..., \alpha_k \in \mathbb{R}$ then $S^-(\alpha_1, ..., \alpha_k)$ denotes the number of strict sign changes in the sequence $\alpha_1, ..., \alpha_k$. If $\lambda \in \tilde{\mathscr{M}}$ and $\operatorname{supp} \lambda \cap [0, 2\pi)$ is finite let

$$\tilde{K}=\operatorname{supp} \lambda=\{x_j : j \in \mathbb{Z}\},$$

where

$$x_j<x_{j+1} \qquad \text{and} \qquad x_{j+m}=x_j+2\pi \qquad \text{for all} \quad j \in \mathbb{Z}.$$

If I is an interval and $\operatorname{supp} \lambda \cap I=\{x_k, ..., x_l\}$ define

$$S^-(\lambda, I)=S^-(\lambda(x_k), ..., \lambda(x_l)).$$

Define

$$\tilde{S}^-(\lambda)=S^-(\lambda(x_1), ..., \lambda(x_{m+1})).$$

Thus $\tilde{S}(\lambda)$ is equal to the number of cyclic sign changes of the measure on the circle which corresponds to the periodic measure λ.

The next theorem will be used repeatedly in Sections 4 and 5.

2.2.2. THEOREM. *Suppose that* $\tilde{K} = \{x_j : j \in \mathbb{Z}\}$, *where*

$$x_j < x_{j+1} \qquad \text{and} \qquad x_{j+m} = x_j + 2\pi \qquad \text{for all} \quad j \in \mathbb{Z},$$

is a periodic set, that $(g, \lambda) \in \mathbb{R} \times \tilde{\mathcal{M}}_0$, $\lambda \neq 0$, *and that* $\operatorname{supp} \lambda \subseteq \tilde{K}$. *Then*

(i) $Z_n(w_n(g, \lambda), (x_j, x_k)) \leqslant S^-(\lambda, (x_j, x_k)) + n$ *whenever* $j < k$;

(ii) $\tilde{Z}_n(w_n(g, \lambda) \leqslant \tilde{S}^-(\lambda)$.

The theorem will be deduced from the results of [4], in particular from the following theorem which follows easily from Corollary 1.7 and Theorem 2.2 of [4].

2.2.3. THEOREM. *Let* $n > 1$. *If* $w \in \mathcal{S}_{n-1}$ *has simple knots at the points*

$$x_1 < \cdots < x_m$$

then

$$Z_n(w, \mathbb{R}) \leqslant S^-(w_+^{(n-1)}(x_1) - w_-^{(n-1)}(x_1), ..., w_+^{(n-1)}(x_m) - w_-^{(n-1)}(x_m)) + n.$$

Now let n, (g, λ) and $\tilde{K}$ be as in the statement of Theorem 2.2.2. If

$$w = w_n(g, \lambda) = g + (-1)^n D_n * \lambda$$

then

$$w^{(n-1)} = (-1)^n D_1 * \lambda$$

and, by Theorem 2.1.1,

$$w_+^{(n-1)}(x) - w_-^{(n-1)}(x) = (-1)^n \lambda(x).$$

Let $w_{j,k} \in \mathcal{S}_{n-1}$ be that non-periodic spline with knots at the points $x_{j+1}, ..., x_{k-1}$ which coincides with w on (x_j, x_k). Then

$$\begin{aligned} Z_n(w, (x_j, x_k)) &= Z_n(w_{j,k}, (x_j, x_k)) \\ &\leqslant Z_n(w_{j,k}, \mathbb{R}) \\ &\leqslant S^-(\lambda(x_{j+1}), ..., \lambda(x_{k-1})) + n \\ &= S^-(\lambda, (x_j, x_k)) + n. \end{aligned}$$

This proves (i) of the theorem.

It is now possible to deduce (ii) from (i). For each $k \in \mathbb{N}$, if $[x_{r-1}, x_r]$ is not a zero interval of w then, by Theorem 2.2.2(i),

$$\begin{aligned}
k\tilde{Z}_n(w) &= kZ_n(w, [x_r, x_{r+m})) \\
&= Z_n(w, [x_r, x_{r+km})) \\
&\leqslant Z_n(w, (x_{r-1}, x_{r+km})) \\
&\leqslant S^-(\lambda, (x_{r-1}, x_{r+km}]) + n \\
&= \sum_{j=0}^{k-1} S^-(\lambda, [x_{r+jm}, x_{r+(j+1)m})) + n \\
&= k\tilde{S}^-(\lambda) + n.
\end{aligned}$$

This holds for all $k \in \mathbb{N}$ and so (ii) is proved.

Theorem 2.2.2(i) will be used repeatedly in the special case in which the function $w_n(g, \lambda)$ has a zero interval, and a zero interval has multiplicity n.

2.2.4. Corollary. *If $w_n(g, \lambda)$ is as in Theorem 2.2.2, $j < k$ and the intervals $[x_{j-1}, x_j]$ and $[x_k, x_{k+1}]$ are zero intervals of $w_n(g, \lambda)$ but $[x_j, x_{j+1}]$ and $[x_{k-1}, x_k]$ are not, then*

$$Z_n(w_n(g, \lambda), (x_j, x_k)) \leqslant S^-(\lambda, (x_{j-1}, x_{k+1})) - n \leqslant k - j - n. \qquad (2.2.1)$$

If $Z_n(w_n(g, \lambda), (x_j, x_k)) = k - j - n$ then $\lambda(x_j), \ldots, \lambda(x_k)$ are all non-zero and they alternate in sign.

3. SEPARATING MEASURES

3.1. *A Preliminary Characterization Theorem*

The set $\tilde{W}_n$ is a proximinal subset of the space $\tilde{C}$, that is, for each $v_0 \in \tilde{C}$ there exists a best uniform approximation u_0 from $\tilde{W}_n$. If $v_0 \in \tilde{C} \backslash \tilde{W}_n$ then the set $\tilde{W}_n$ and the open ball $\{v \in \tilde{C}: \|v - v_0\| < d(v_0, \tilde{W}_n)\}$ are convex and disjoint and can be separated by a non-zero linear functional $\lambda \in \tilde{C}^* \simeq \mathscr{M}$. Let $\phi: \tilde{C} \to C(T)$ be the natural isometric isomorphism of $\tilde{C}$ onto the space $C(T)$ of continuous real valued functions on the circle T. The starting point for our analysis of best approximations from $\tilde{W}_n$ is a general result of Brown ([3, Theorem 1]) which, applied to the subset $\phi(\tilde{W}_n)$ of $C(T)$ and translated from $C(T)$ to $\tilde{C}$, contains the following preliminary characterization theorem as a special case.

3.1.1. THEOREM. *Let* $n \in \mathbb{N}$. *Suppose* $v_0 \in \tilde{C} \backslash \tilde{W}_n$, $u_0 \in \tilde{W}_n$, $\lambda \in \tilde{\mathscr{M}} \backslash \{0\}$, *and* $g \in \mathbb{R}$, *and let*

$$w_n(g, \lambda) = g + (-1)^n D_n * \lambda. \tag{3.1.1}$$

Then the three conditions

I(a). u_0 *is a best approximation to* v_0 *from* $\tilde{W}_n$,

I(b). $\lambda(u) < \lambda(v)$ *for all* $u \in \tilde{W}_n$ *and all* $v \in \tilde{C}$ *such that* $\|v_0 - v\| < d(v_0, \tilde{W}_n)$.

I(c). $\|w_n(g, \lambda)\|_1 \leqslant \|w_n(g, \lambda) - h\|_1$ *for all* $h \in \mathbb{R}$ (*that is,* $-g$ *is a best* L^1*-approximation to* $(-1)^n D_n * \lambda$ *from the space of constant functions*), *are together equivalent to the three conditions*

II(i). $\lambda([0, 2\pi)) = 0$,

II(ii). $u_0^{(n)}(y) = \operatorname{sgn} w_n(g, \lambda)(y)$ *for almost every* y *in* $\mathbb{R} \backslash w_n(g, \lambda)^{-1}(0)$,

II(iii). $\lambda(v_0 - u_0) = \|\lambda\| \, \|v_0 - u_0\|$ *or, equivalently,*

$$\operatorname{supp} \lambda^+ \subseteq (v_0 - u_0)^{-1}(\|v_0 - u_0\|),$$
$$\operatorname{supp} \lambda^- \subseteq (v_0 - u_0)^{-1}(-\|v_0 - u_0\|).$$

Terminology. If $v_0 \in \tilde{C} \backslash \tilde{W}_n$ then a measure $\lambda \in \tilde{\mathscr{M}}$ which satisfies condition I(b) will be called a *separating measure* for v_0 and $\tilde{W}_n$. Thus, by condition II(i), separating measures lie in $\tilde{\mathscr{M}}_0$. Let $S(v_0, \tilde{W}_n)$ denote the set of pairs $(g, \lambda) \in \mathbb{R} \times \tilde{\mathscr{M}}$ such that λ is a separating measure for v_0 and $\tilde{W}_n$, and g and λ satisfy condition I(c).

Note that if λ is a separating measure for v_0 and $\tilde{W}_n$ then, by condition I(iii), $\|\lambda\| = 1$ if and only if $\|\lambda\| \leqslant 1$ and $\lambda(v_0 - u_0) = \|v_0 - u_0\|$. It follows that $\{(g, \lambda) \in S(v_0, \tilde{W}_n) : \|\lambda\| = 1\}$ is a compact subset of $\mathbb{R} \times \tilde{\mathscr{M}}$ in the product topology obtained from the weak*-topology of $\tilde{C}^* \cong \tilde{\mathscr{M}}$.

If $(g, \lambda) \in S(v_0, \tilde{W}_n)$ then the function $w_n(g, \lambda)$ will be called *an associated function* of v_0 and $\tilde{W}_n$ or the *associated function* of (g, λ). If $n > 1$ the associated functions are continuous.

The theorem has an immediate and significant corollary.

3.1.2. COROLLARY. *If* w *is an associated function of* v_0 *and* $\tilde{W}_n$ *and* $w^{-1}(0)$ *is a null set then there is a unique best approximation to* v_0 *from* $\tilde{W}_n$.

Proof. Suppose that w is an associated function of v_0 and $\tilde{W}_n$ and that u_0 is a best approximation to v_0 from $\tilde{W}_n$. If $w^{-1}(0)$ is null then, by condition II(ii), $u_0^{(n)}(y) = \operatorname{sgn} w(y)$ for almost all $y \in \mathbb{R}$. Now $u_0 = c + D_n * u_0^{(n)}$ for some $c \in \mathbb{R}$. It follows that c is the unique best uniform approximation from the space of constant functions to $v_0 - D_n * \operatorname{sgn} w$.

In several proofs it will be necessary, given one element of $S(v_0, \tilde{W}_n)$, to construct another. The next proposition, which is a straightforward consequence of Theorem 3.1.1, gives conditions which are sufficient to ensure that a pair (g, λ) is an element of $S(v_0, \tilde{W}_n)$.

3.1.3. PROPOSITION. *Suppose* $v_0 \in \tilde{C} \backslash \tilde{W}_n$ *and* $(g_0, \lambda_0) \in S(v_0, \tilde{W}_n)$. *If* $(g, \lambda) \in \mathbb{R} \in \tilde{\mathscr{M}}_0$,

$$\operatorname{supp} \lambda^+ \subseteq \operatorname{supp} \lambda_0^+, \qquad \operatorname{supp} \lambda^- \subseteq \operatorname{supp} \lambda_0^-, \tag{3.1.2}$$

$$w_n(g, \lambda)^{-1}(0) \supseteq w_n(g_0, \lambda_0)^{-1}(0), \tag{3.1.3}$$

and

$$w_n(g, \lambda)(y)\, w_n(g_0, \lambda_0)(y) \geqslant 0 \qquad \text{for all} \quad y \in \mathbb{R}, \tag{3.1.4}$$

then (g, λ) *is also an element of* $S(v_0, \tilde{W}_n)$.

Proof. If $(g_0, \lambda_0) \in S(v_0, \tilde{W}_n)$ then g_0 and λ_0 satisfy conditions II(i)–(iii) of Theorem 3.1.1. It follows from the conditions (3.1.2)–(3.1.4) that g and λ also satisfy conditions II(i)–(iii). So the conclusion follows by Theorem 3.1.1.

The central results of [3] concerning best approximation from W_n in $C([0, 1])$ have as a corollary the fact that if $v_0 \in C([0, 1]) \backslash W_n$ then there exists a separating measure $\lambda \in C([0, 1])^*$ for v_0 and W_n which has finite support. It was suggested in [3] that a direct proof of this fact might yield greater insight into the general problem. This expectation is confirmed by our development. In this section it is first established that there exist separating measures with minimal supports and associated functions with maximal zero sets.

3.1.4. THEOREM. *Suppose* $v_0 \in \tilde{C} \backslash \tilde{W}_n$. *If* λ_1 *is a separating measure for* v_0 *and* $\tilde{W}_n$ *then there exists a separating measure* λ_0 *such that* $\operatorname{supp} \lambda_0 \subseteq \operatorname{supp} \lambda_1$ *and if* λ *is also a separating measure and* $\operatorname{supp} \lambda \subseteq \operatorname{supp} \lambda_0$ *then* $\operatorname{supp} \lambda = \operatorname{supp} \lambda_0$.

Proof. For each λ which is a separating measure for v_0 and $\tilde{W}_n$ let $L(\lambda)$ denote the set of separating measures λ' such that $\|\lambda'\| = 1$ and $\operatorname{supp} \lambda' \subseteq \operatorname{supp} \lambda$. Then $L(\lambda)$ is a non-empty weak*-compact subset of $\tilde{C}^* \cong \tilde{\mathscr{M}}$. Therefore any chain of sets of the form $L(\lambda)$ has a non-empty intersection. It follows that for each separating measure λ_1 there is a subset $L(\lambda_0)$ of $L(\lambda_1)$ which is minimal amongst all sets of this form. This proves the theorem.

3.1.5. THEOREM. *Suppose* $n > 1$. *Let* $v_0 \in \tilde{C} \backslash \tilde{W}_n$ *and let* λ_1 *be a separating measure for* v_0 *and* $\tilde{W}_n$ *with minimal support. Then there exists* $(g_0, \lambda_0) \in S(v_0, \tilde{W}_n)$ *such that*

(i) $\operatorname{supp} \lambda_0 = \operatorname{supp} \lambda_1$, *and*

(ii) $w_n(g, \lambda)^{-1}(0) = w_n(g_0, \lambda_0)^{-1}(0)$ *wherever* $(g, \lambda) \in S(v_0, \tilde{W}_n)$, $\operatorname{supp} \lambda = \operatorname{supp} \lambda_0$, *and* $w_n(g, \lambda)^{-1}(0) \supseteq w_n(g_0, \lambda_0)^{-1}(0)$.

Proof. For each $(g, \lambda) \in S(v_0, \tilde{W}_n)$ let $L(g, \lambda)$ denote the set of $(g', \lambda') \in S(v_0, \tilde{W}_n)$ such that $\|\lambda'\| = 1$, $\operatorname{supp} \lambda' \subseteq \operatorname{supp} \lambda$, and

$$w_n(g', \lambda')^{-1}(0) \supseteq w_n(g, \lambda)^{-1}(0).$$

If $n > 1$ then each of the sets $L(g, \lambda)$ is a non-empty and compact subset of $\mathbb{R} \times \mathscr{M}$. (However, if $n = 1$ then the mapping $(g, \lambda) \mapsto w_n(g, \lambda)(y)$ is not continuous.) The completion of the proof now follows that of the previous theorem.

The lemma which follows provides the crucial step in proving that a separating measure with minimal support corresponds to a measure on the circle with finite support.

3.1.6. LEMMA. *Suppose* $n > 1$. *Let* $v_0 \in \tilde{C} \backslash \tilde{W}_n$ *and let* $(g_0, \lambda_0) \in S(v_0, \tilde{W}_n)$ *be such that* λ_0 *has minimal support and* $w_n(g_0, \lambda_0)$ *has maximal zero set, that is, satisfies the condition* (ii) *of Theorem* 3.1.5. *If* $a < b < a + 2\pi$ *and* $w_n(g_0, \lambda_0)^{-1}(0) \cap [a, b] = \varnothing$ *then* $|\operatorname{supp} \lambda_0 \cap [a, b]| \leqslant n + 1$.

Proof. The condition that $n > 1$ ensures that the associated functions are continuous. The lemma will be proved by contradiction. Suppose that $|\operatorname{supp} \lambda_0 \cap [a, b]| > n + 1$. Let $B_1, B_2, ..., B_{n+1}, B_{n+2}$ be disjoint closed subintervals of (a, b) centred upon $n + 2$ distinct points of $\operatorname{supp} \lambda_0 \cap (a, b)$. Let $\lambda_j \in \tilde{\mathscr{M}}$, for $j = 1, ..., n + 1$ (but not for $n + 2$), be the periodic measure such that $\lambda_j(A) = \lambda_0(A \cap B_j)$ for each Borel subset A of $[a, a + 2\pi)$.

Let $y_1, ..., y_n$ be distinct points of $(b, a + 2\pi)$. Then there exists a nonzero $(g, a_1, ..., a_{n+1}) \in \mathbb{R}^{n+2}$ such that

$$(a_1 \lambda_1 + \cdots + a_{n+1} \lambda_{n+1})([a, a + 2\pi)) = 0$$

and

$$w_n(g, a_1 \lambda_1 + \cdots + a_{n+1} \lambda_{n+1})(y_k) = 0$$

for $k = 1, ..., n$. Let $\lambda = a_1 \lambda_1 + \cdots + a_{n+1} \lambda_{n+1}$. So $\lambda \neq 0$ and $\lambda \in \tilde{\mathscr{M}}_0$. Now $\operatorname{supp} \lambda \cap (b, a + 2\pi) = \varnothing$ so, by Proposition 2.1.5, the restriction of $w_n(g, \lambda)$ to the interval $(b, a + 2\pi)$ is a polynomial of degree $\leqslant n - 1$ with n zeros in $(b, a + 2\pi)$. Thus $(b, a + 2\pi)$ is a zero interval of $w_n(g, \lambda)$.

Let $\varepsilon \in \{-1, 1\}$ be the sign of $w_n(g_0, \lambda_0)$ on $[a, b]$. Let J be the set of $a \in \mathbb{R}$ such that

$$aa_j > -1 \qquad \text{for} \quad j = 1, ..., n+1,$$

and

$$\varepsilon w_n(g_0 + ag, \lambda_0 + a\lambda)(y) > 0 \qquad \text{for all} \quad y \in [a, b].$$

Then $0 \in J \neq \mathbb{R}$ and J is an open subinterval of $\mathbb{R}$.

Suppose $a \in J^-$. Then $\operatorname{supp}(\lambda_0 + a\lambda) \supseteq \operatorname{supp} \lambda_0 \cap B_{n+2} \neq \varnothing$ so that $\lambda_0 + a\lambda \neq 0$; $\lambda_0 + a\lambda \in \tilde{\mathcal{M}}_0$; and

$$\operatorname{supp}(\lambda_0 + a\lambda)^+ \subseteq \operatorname{supp} \lambda_0^+,$$

$$\operatorname{supp}(\lambda_0 + a\lambda)^- \subseteq \operatorname{supp} \lambda_0^-.$$

Furthermore

$$w_n(g_0 + ag, \lambda_0 + a\lambda) = w_n(g_0, \lambda_0) + aw_n(g, \lambda)$$

and it follows that

$$w_n(g_0 + ag, \lambda_0 + a\lambda)^{-1}(0) \supseteq w_n(g_0, \lambda_0)^{-1}(0)$$

and

$$w_n(g_0 + ag, \lambda_0 + a\lambda)(y)\, w_n(g_0, \lambda_0)(y) \geqslant 0 \qquad \text{for all} \quad y \in \mathbb{R}.$$

Therefore, by Proposition 3.1.3, $(g_0 + ag, \lambda_0 + a\lambda) \in S(v_0, \tilde{W}_n)$.

Now let a be a point of the non-empty boundary of J. Then either $aa_j = -1$ for some $j \in \{1, ..., n+1\}$, in which case $\operatorname{supp}(\lambda_0 + a\lambda) \cap B_j = \varnothing$ and $\operatorname{supp}(\lambda_0 + a\lambda) \neq \operatorname{supp} \lambda_0$, in contradiction to the fact that λ_0 is a separating measure of minimal support, or $w_n(g_0 + ag, \lambda_0 + a\lambda)(y) = 0$ for some $y \in [a, b]$, in contradiction to the fact that $w_n(g_0, \lambda_0)$ has maximal zero set. The proof of the lemma is complete.

The "finite support theorem" now follows easily.

3.1.7. THEOREM. *Suppose $n > 1$. If λ is a separating measure of minimal support for some $v_0 \in \tilde{C} \backslash \tilde{W}_n$ then* $\operatorname{supp} \lambda \cap [0, 2\pi)$ *is finite.*

Proof. By Theorem 3.1.5 it may be supposed (after replacing λ by another measure with the same support if necessary) that there is a $g \in \mathbb{R}$ such that $(g, \lambda) \in S(v_0, \tilde{W}_n)$ and the associated function $w_n(g, \lambda)$ has maximal zero set. It follows from (ii) and (iii) of Theorem 2.1.5 that $w_n(g, \lambda)$ is a nonzero piecewise monotonic function. Therefore $w_n(g, \lambda)^{-1}(0)$ is a union of a finite family of zero intervals and a finite set of isolated points. If I is a zero interval of $w_n(g, \lambda)$ then, by Theorem 2.1.5, $\operatorname{supp} \lambda \cap \operatorname{int} I$ is empty. If J is an open interval of $[0, 2\pi)$ disjoint from $w_n(g, \lambda)^{-1}(0)$ then it follows from Lemma 3.1.6 that $|\operatorname{supp} \lambda \cap J| < n+2$. This proves the theorem.

4. ON THE KNOTS AND ZEROS OF PERIODIC SPLINES

4.1. *Introduction*

Suppose $n > 1$. If $v_0 \in \tilde{C} \setminus \tilde{W}_n$, u_0 is a best approximation to v_0 from $\tilde{W}_n$, $(g, \lambda) \in S(v_0, \tilde{W}_n)$, and λ is a separating measure with minimal support then, by Theorem 3.1.7, $\operatorname{supp} \lambda \cap [0, 2\pi)$ is finite and the associated function $w_n(g, \lambda)$ is a periodic spline of degree $n-1$ with simple knots at the points of $\operatorname{supp} \lambda$. By Theorem 3.1.1 the knots are points at which the function $v_0 - u_0$ attains its norm and the points at which $w_n(g, \lambda)$ changes sign are points of discontinuity of $u_0^{(n)}$. The main result of the paper, the characterization of best approximations to v_0 from $\tilde{W}_n$, will depend upon an analysis of the relation between the knots, zeros, and signs of periodic splines—not in complete generality but subject to certain restrictions.

There are two mutually exclusive cases to be considered:

I. There exists $(g, \lambda) \in S(v_0, \tilde{W}_n)$ such that $\operatorname{supp} \lambda \cap [0, 2\pi)$ is finite and the associated function $w_n(g, \lambda)$ has a zero interval, and

II. If $(g, \lambda) \in S(v_0, \tilde{W}_n)$ and λ has minimal support then the associated function $w_n(g, \lambda)$ has no zero interval.

Theorem 4.2.2 gives conditions which guarantee the existence of a spline with specified knots, zeros, and zero interval. Lemmas 4.3.5 and 4.3.7 exhibit splines with specified knots and zeros (subject to certain restrictions), but no zero interval.

4.1.1. *Notation.* The following notation will be used throughout this section. Let $m, q \in \mathbb{N}$. Let $\tilde{K} = \{x_j; j \in \mathbb{Z}\}$ be a periodic sequence such that

$$x_1 < \cdots < x_m < x_1 + 2\pi \qquad \text{and} \qquad x_{j+m} = x_j + 2\pi \quad \text{for all} \quad j \in \mathbb{Z},$$

and let $\tilde{Z} = \{z_i \colon i \in \mathbb{Z}\}$ be a periodic sequence such that

$$z_1 < \cdots < z_q < z_1 + 2\pi \qquad \text{and} \qquad z_{i+q} = z_i + 2\pi \quad \text{for all} \quad i \in \mathbb{Z}.$$

For each $k \in \mathbb{Z}$ let $h(k)$ denote the integer such that $x_k \in (z_{h(k)-1}, z_{h(k)}]$.

The set $\tilde{K}$ will be a set of actual or potential knots and $\tilde{Z}$ will be a set of actual or potential zeros of a given spline function or of a spline function which has to be constructed. The set $\tilde{K}$ corresponds to a set of m points on the circle, $\tilde{Z}$ to a set of q points on the circle. The number of elements of a finite set A will be denoted $|A|$. Thus $|\tilde{K} \cap [0, 2\pi)| = m$.

Let $P_n(\tilde{K}, \tilde{Z})$ denote the set of pairs $(g, \lambda) \in \mathbb{R} \times \tilde{\mathscr{M}}_0$ such that supp $\lambda \subseteq \tilde{K}$ and $\tilde{Z} \subseteq w_n(g, \lambda)^{-1}(0)$.

4.2. *Splines with Zero Intervals*

4.2.1. LEMMA. *Suppose* $(g, \lambda) \in \mathbb{R} \times \tilde{\mathscr{M}}_0$, $a < b < c$, supp $\lambda \cap (a, b) = \varnothing$, *and* $(b, c) \subseteq w_n(g, \lambda)^{-1}(0)$. *Then* supp $\lambda \cap (b, c) = \varnothing$ *and*

$$w_n(g, \lambda)(y) = \lambda(b) \frac{(b-y)^{n-1}}{(n-1)!}$$

for all $y \in (a, b)$.

Proof. By 2.1.5, supp $\lambda \cap (b, c) = \varnothing$. Then, by 2.1.1,

$$w_n(g, \lambda)^{(n-1)}(y) = (-1)^n (D_1 * \lambda)(y) = \begin{cases} (-1)^{n+1} \lambda(b) & \text{for} \quad y \in (a, b) \\ 0 & \text{for} \quad y \in (b, c)' \end{cases}$$

and the assertion of the lemma follows.

The next theorem establishes a sufficient condition for the existence of a spline with specified knots and zeros and with a zero interval.

4.2.2. THEOREM. *Let* $n > 1$. *Let* $m, q, \tilde{K}, \tilde{Z}$, *and* h *be as in paragraph* 4.1.1. *Suppose that there exist integers* j, k *such that* $j < k < j + m$,

$$|\tilde{Z} \cap (x_j, x_k)| \leqslant k - j - n, \tag{4.2.1}$$

and

$$|\tilde{Z} \cap (x_{j'}, x_{k'})| > k' - j' - n \tag{4.2.2}$$

whenever $j \leqslant j' < k' \leqslant k$ *and* $(j', k') \neq (j, k)$. *Then* $|\tilde{Z} \cap (x_j, x_k)| = k - j - n$ *and there exists* $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$ *such that*

$$[x_k, x_j + 2\pi] \subseteq w_n(g, \lambda)^{-1}(0) \tag{4.2.3}$$

$$(-1)^{i + h(k)} \lambda(x_k) w_n(g, \lambda)(y) > 0 \tag{4.2.4}$$

for all $y \in (z_{i-1}, z_i) \cap (x_j, x_k)$ *and* $i \in \mathbb{Z}$, *and*

$$(-1)^{j' + k} \lambda(x_k) \lambda(x_{j'}) > 0 \tag{4.2.5}$$

for $j' = j, ..., k$.

Proof. Suppose that $|\tilde{Z} \cap (x_j, x_k)| < k - j - n$. Then $k - j > n \geqslant 2$ and $|\tilde{Z} \cap (x_j, x_{k-1})| \leqslant (k-1) - j - n$ in contradiction to condition (4.2.2). Therefore $|\tilde{Z} \cap (x_j, x_k)| = k - j - n$.

Let $y_1, ..., y_n$ be distinct points of $(x_k, x_j + 2\pi)$. Let δ_x denote the periodic measure with support $x + 2\pi\mathbb{Z}$ and such that $\delta_x(x) = 1$. Then there exists a non-zero $(g, a_j, ..., a_k) \in \mathbb{R}^{k-j+2}$ such that

$$a_j + \cdots + a_k = 0,$$

and, if $\lambda = a_j \delta_{x_j} + \cdots + a_k \delta_{x_k}$ then

$$w_n(g, \lambda)(y_i) = 0 \qquad \text{for} \quad i = 1, ..., n$$
$$w_n(g, \lambda)(z) = 0 \qquad \text{for} \quad z \in \tilde{Z} \cap (x_j, x_k).$$

Then $\lambda \in \mathscr{M}_0 \backslash \{0\}$. Also, $\operatorname{supp} \lambda \cap [x_j, x_j + 2\pi] \subseteq \{x_j, ..., x_k\}$ so the restriction of $w_n(g, \lambda)$ to $[x_k, x_j + 2\pi]$ is a polynomial of degree $n-1$ with n zeros and therefore condition (4.2.3) is satisfied.

Let j', k' be the integers such that $j \leqslant j' < k' \leqslant k$ and

$$\{x_{j'}, x_{k'}\} \subseteq \operatorname{supp} \lambda \cap [x_j, x_j + 2\pi) \subseteq \{x_{j'}, ..., x_{k'}\}.$$

Then $w_n(g, \lambda)(y) = 0$ for all $y \in [x_{k'}, x_{j'} + 2\pi]$. Then, by Corollary 2.2.4,

$$\begin{aligned} k' - j' - n &\leqslant |\tilde{Z} \cap (x_{j'}, x_{k'})| \\ &\leqslant Z_n(w_n(g, \lambda), (x_{j'}, x_{k'})) \\ &\leqslant S^-(\lambda, [x_{j'}, x_{k'}]) - n \\ &\leqslant k' - j' - n. \end{aligned}$$

So the preceding inequalities are all equalities and they entail that $(j', k') = (j, k)$ (by condition (4.2.2)), that $w_n(g, \lambda)^{-1}(0) \cap (x_j, x_k) = \tilde{Z} \cap (x_j, x_k)$, that $w_n(g, \lambda)$ changes sign at each of its zeros in (x_j, x_k) (by 2.2.1), and that $S^-(\lambda, [x_j, x_k]) = k - j$. The inequalities (4.2.5) now follow. The sign of $w_n(g, \lambda)$ to the left of x_k is determined by the sign of $\lambda(x_k)$, according to Lemma 4.2.1, and the inequalities (4.2.4) follow.

The next theorem shows that the interlacing condition (4.2.1) of Theorem 4.2.2, satisfied by the knots $\tilde{K}$ and zeros $\tilde{Z}$, is also a necessary condition. This condition—usually in the form of its negation—plays a central role in everything which follows.

4.2.3. THEOREM. *Let $n > 1$. Let $m, q, \tilde{K}, \tilde{Z}$ be as in* 4.1.1.

If $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$, $w_n(g, \lambda) \neq 0$, and $w_n(g, \lambda)$ has a zero interval then $\lambda \neq 0$ and there exist integers j, k such that $j < k < j + m$,

$$w_n(g, \lambda)(y) = 0 \qquad \textit{for all} \quad y \in [x_{j-1}, x_j] \cup [x_k, x_{k+1}], \qquad (4.2.6)$$

and

$$|\tilde{Z} \cap (x_j, x_k)| \leqslant k - j - n.$$

Proof. Suppose that (g, λ) satisfies the stated conditions. Then, by 2.1.5, $\lambda \neq 0$. Also, $w_n(g, \lambda) \in \mathscr{S}_{n-1}$. There exist j, k such that $j < k$, the condition (4.2.6) is satisfied, and (x_j, x_k) contains no zero interval of $w_n(g, \lambda)$. So $x_k < x_j + 2\pi$ and $j < k < j + m$. Then, appealing to Corollary 2.2.4, it follows that

$$\begin{aligned} |\tilde{Z} \cap (x_j, x_k)| &\leqslant |w_n(g, \lambda)^{-1}(0) \cap (x_j, x_k)| \\ &\leqslant Z_n(w_n(g, \lambda), (x_j, x_k)) \\ &\leqslant S^-(\lambda, [x_j, x_k]) - n \\ &\leqslant k - j - n. \end{aligned}$$

Theorem 4.2.2 will be invoked in situations in which the hypotheses of the theorem are not immediately satisfied. The following technical and combinatorial lemma is required as a bridge from the circumstance which will be considered to the hypotheses of the theorem.

4.2.4. LEMMA. *Let $m, q, \tilde{K}, \tilde{Z}$, and h be as in paragraph* 4.1.1 *and let* $\varepsilon_M, \varepsilon_F \in \{-1, 1\}$.

Suppose that there exist integers j, k such that $j < k < j + m$ and the following condition (C) *is satisfied: either*

$$|\tilde{Z} \cap (x_j, x_k)| < k - j - n \tag{C1}$$

or

$$|\tilde{Z} \cap (x_j, x_k)| = k - j - n \quad \text{and} \quad (-1)^{q - h(k)} \varepsilon_F = (-1)^{m-k} \varepsilon_M. \tag{C2}$$

Then if (j, k) is a minimal interval of integers such that condition (C) *is satisfied it follows that*

$$|\tilde{Z} \cap (x_j, x_k)| = k - j - n \tag{4.2.7}$$

and

$$|\tilde{Z} \cap (x_{j'}, x_{k'})| > k' - j' - n$$

whenever $j \leqslant j' < k' \leqslant k$ and $(j', k') \neq (j, k)$.

Proof. Let (j, k) be a minimal interval of integers such that condition (C) is satisfied.

Suppose that $|\tilde{Z} \cap (x_j, x_k)| < k - j - n$. It will be shown that this is incompatible with the minimality of (j, k).

If $(-1)^{q-h(k)} \varepsilon_F = (-1)^{m-k} \varepsilon_M$ then $|\tilde{Z} \cap (x_{j+1}, x_k)| \leqslant k - (j+1) - n$ and condition (C) is satisfied by the interval $(j+1, k)$. If $\tilde{Z} \cap [x_{k-1}, x_k) \neq \emptyset$ then $|\tilde{Z} \cap (x_j, x_{k-1})| < (k-1) - j - n$ and the condition (C) is satisfied by $(j, k-1)$. If $\tilde{Z} \cap [x_{k-1}, x_k) = \emptyset$ and $(-1)^{q-h(k)} \varepsilon_F = -(-1)^{m-k} \varepsilon_M$ then $h(k-1) = h(k)$ and $(-1)^{q-h(k-1)} = (-1)^{m-(k-1)}$ so that condition (C) is satisfied by the interval $(j, k-1)$. This completes the proof of Eq. (4.2.7) and it follows that (j, k) satisfies condition (C2).

If $j < j' < k$ then (j', k) does not satisfy condition (C) and so

$$|\tilde{Z} \cap (x_{j'}, x_k)| > k - j' - n,$$

from which it follows that

$$|\tilde{Z} \cap (x_j, x_{j'}]| = |\tilde{Z} \cap (x_j, x_k)| - |\tilde{Z} \cap (x_{j'}, x_k)| < j' - j.$$

Let $j < k' < k$. Then (j, k') does not satisfy condition (C). Suppose that $|\tilde{Z} \cap (x_j, x_{k'})| = k' - j - n$. Then $|\tilde{Z} \cap [x_{k'}, x_k)| = k - k'$ and it follows that $h(k') = h(k) - (k - k')$ and that (j, k') satisfies condition (C2) which is a contradiction. Therefore

$$|\tilde{Z} \cap (x_j, x_{k'})| > k' - j - n,$$

and it follows that

$$|\tilde{Z} \cap [x_{k'}, x_k)| < k - k'.$$

Now, if $j < j' < k' < k$ then

$$\begin{aligned} |\tilde{Z} \cap (x_{j'}, x_{k'})| &= |\tilde{Z} \cap (x_j, x_k)| - |\tilde{Z} \cap (x_j, x_{j'}]| - |\tilde{Z} \cap [x_{k'}, x_k)| \\ &> (k - j - n) - (j' - j) - (k - k') = k' - j' - n. \end{aligned}$$

The proof of the lemma is complete.

The following technical proposition shows that Condition I of Theorem 5.2.1 is equivalent to the condition of Sattes's characterisation theorem for best approximations from W_n to a function in $C([0, 1])$.

4.2.5. Proposition. *Suppose that* $[a, b]$, q, m, ε_M *and the points*

$$a = z_1 < \cdots < z_q = b$$

and

$$a = x_1 < \cdots < x_m = b$$

are as in Condition I of Theorem 5.2.1 *and let* $\tilde{K}$ *and* $\tilde{Z}$ *be the periodic extensions of the sets* $\{x_1, ..., x_m\}$ *and* $\{z_1, ..., z_q\}$ *respectively. Then* $(1, m)$ *is a minimal interval of integers satisfying condition* (C) *of Lemma* 4.2.4, *with* $\varepsilon_F = \varepsilon_M$, *if and only if*

$$x_j < z_j < x_{j-1+n} \qquad \text{for} \quad j = 2, ..., m-n. \tag{4.2.8}$$

Proof. If $j \in \{2, ..., m-n\}$ then

$$x_j < z_j \qquad \text{if and only if} \quad |\tilde{Z} \cap (x_j, x_k)| > m - j - n. \tag{4.2.9}$$

If $k \in \{2, ..., m-n\}$ then

$$z_k < x_{k-1+n} \qquad \text{if and only if} \quad |\tilde{Z} \cap (x_1, x_k)| > k - 1 - n. \tag{4.2.10}$$

Suppose that $1, m$ is a closest pair of integers satisfying condition (C) with $\varepsilon_F = \varepsilon_M$.

If $j \in \{2, ..., m-n\}$ then the interval of integers (j, m) does not satisfy condition (C) and so $|\tilde{Z} \cap (x_j, x_m)| > m - j - n$. So, by (4.2.9), $x_j < z_j$.

If $k \in \{2, ..., m-n\}$ then the interval of integers $(1, k)$ does not satisfy condition (C). However, if $|\tilde{Z} \cap (x_1, x_k)| = k - 1 - n$ then $\tilde{Z} \cap (x_1, x_k) = \{z_2, ..., z_{k-n}\}$ and $h(k) = k - n + 1$ so that condition (C2) is satisfied. Therefore $|\tilde{Z} \cap (x_1, x_k)| > k - 1 - n$ and so, by (4.2.10), $z_k < x_{k-1+n}$.

Conversely, suppose the inequalities (4.2.8) are satisfied. Then, by (4.2.9) and (4.2.10), the intervals of integers (j, m) for $j \in \{2, ..., m-n\}$ and $(1, k)$ for $k \in \{2, ..., m-n\}$ do not satisfy condition (C). Also, if $1 < j < k < m$ then

$$\begin{aligned} |\tilde{Z} \cap (x_j, x_k)| &= |\tilde{Z} \cap (x_1, x_k)| + |\tilde{Z} \cap (x_j, x_m)| - |\tilde{Z} \cap (x_1, x_m)| \\ &> k - j - n, \end{aligned}$$

and the interval of integers (j, k) does not satisfy condition (C). This proves that $(1, m)$ is a minimal interval of integers satisfying condition (C). The proof of the proposition is complete.

4.3. *Periodic Splines without Zero Intervals*

4.3.1. *Notation.* If $m = q$ then we will write

$$\mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix} = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & D_n(x_1 - z_1) & \cdots & D_n(x_m - z_1) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & D_n(x_1 - z_q) & \cdots & D_n(x_m - z_q) \end{vmatrix}. \tag{4.3.1}$$

These determinants, with this notation, were considered by Brown [2]. The convolution kernels D_n have a sign-regularity property which was established in [2]; it is stated (for $n>1$) as the first part of the next theorem.

4.3.2. THEOREM. *Let* $n>1$. *Suppose that* $m, q, \tilde{K}, \tilde{Z}$ *are as in paragraph* 4.1.1, *that* q *is an odd integer, and* $m=q$.

(i) [2]

$$\mathscr{D}_n\begin{pmatrix}1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q\end{pmatrix} \leqslant 0.$$

(ii)
$$\mathscr{D}_n\begin{pmatrix}1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q\end{pmatrix} = 0,$$

if and only if there exist j, k *such that* $j<k<j+m$ *and*

$$|\tilde{Z} \cap (x_j, x_k)| \leqslant k-j-n. \tag{4.3.2}$$

Proof. Let

$$D=\mathscr{D}_n\begin{pmatrix}1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q\end{pmatrix}.$$

Then $D=0$ if and only if the columns of the matrix in (4.3.1) are linearly dependent. The latter condition is equivalent to the existence of a non-zero $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$. Suppose that $D=0$ and that (g, λ) is a non-zero element of $P_n(\tilde{K}, \tilde{Z})$. Suppose, contrary to the assertion of (ii), that

$$|\tilde{Z} \cap (x_j, x_k)| > k-j-n$$

whenever $j<k<j+m$. Then, by Theorem 4.2.3, $w_n(g, \lambda)$ has no zero interval. Now m is an odd integer and $\tilde{S}^-(\lambda)$ is an even integer. So, by Theorem 2.2.2(ii),

$$\begin{aligned} m > \tilde{S}^-(\lambda) &\geqslant \tilde{Z}_n(w_n(g, \lambda)) = Z_n(w_n(g, \lambda), [0, 2\pi)) \\ &\geqslant |w_n(g, \lambda)^{-1}(0) \cap [0, 2\pi)| \geqslant |\tilde{Z} \cap [0, 2\pi)| = m, \end{aligned}$$

and we have obtained a contradiction.

Suppose that there exist j, k such that $j<k<j+m$ and the inequality (4.3.2) is satisfied. Let j, k be a closest such pair of integers. Then by Theorem 4.2.2 there exists a non-zero $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$ and so $D=0$. The proof of (ii) is complete. (It is also possible to prove (ii) by a refinement of the argument by which (i) is proved in [2].)

If $m = q+1$ and we expand the determinant

$$\mathscr{D}_n\begin{pmatrix} 1; & x_1 \cdots x_{m-1} & x_m \\ 1; & z_1 \cdots z_q & y \end{pmatrix}$$

by the last row of the matrix we obtain the following proposition.

4.3.3. PROPOSITION. *Suppose $m = q+1$. Then for all $y \in \mathbb{R}$,*

$$\mathscr{D}_n\begin{pmatrix} 1; & x_1 \cdots x_{m-1} & x_m \\ 1; & z_1 \cdots z_q & y \end{pmatrix} = w_n(g, \lambda)(y),$$

where $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$ and, for $j = 1, ..., m$,

$$\lambda(x_j) = (-1)^{m+j}\,\mathscr{D}_n\begin{pmatrix} 1; & x_1 \cdots x_{j-1} & x_{j+1} \cdots x_m \\ 1; & z_1 \cdots & \cdots \quad \cdots z_q \end{pmatrix}.$$

4.3.4. *Assumptions.* In the next three lemmas it is assumed that $n > 1$, that q is an even integer, that $\tilde{K}$ and $\tilde{Z}$ are as in paragraph 4.1.1, and that

$$|\tilde{Z} \cap (x_j, x_k)| > k - j - n \qquad \text{whenever} \quad j < k < j + m. \tag{4.3.3}$$

4.3.5. LEMMA. *Let $m = q+1$. If*

$$\mathscr{D}_n\begin{pmatrix} 1; & x_1 \cdots x_{m-1} & x_m \\ 1; & z_1 \cdots z_q & y \end{pmatrix} = w_n(g, \lambda)(y), \qquad \text{for all} \quad y \in \mathbb{R},$$

where $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$, then $\lambda \neq 0$.

Proof. It follows from the assumption (4.3.3) and Theorem 4.3.2(ii) that $w_n(g, \lambda) \neq 0$ and so $\lambda \neq 0$.

Note that $\dim P_n(\tilde{K}, \tilde{Z}) \geqslant 1$ if either $m > q$ or $m = q$ and

$$\mathscr{D}_n\begin{pmatrix} 1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix} = 0.$$

4.3.6. LEMMA. *Let $n > 1$. Suppose that $m \in \{q, q+1\}$.*

(i) *If (g, λ) is a non-zero element of $P_n(\tilde{K}, \tilde{Z})$ then*

$$w_n(g, \lambda)^{-1}(0) = \tilde{Z}, \tag{4.3.4}$$

there exists a sign $\varepsilon_F \in \{-1, 1\}$ such that

$$(-1)^i\,\varepsilon_F w_n(g, \lambda)(y) > 0 \tag{4.3.5}$$

for all $y \in (z_{i-1}, z_i)$ *and* $i \in \mathbb{Z}$, *and, for some* $a \in \{1, ..., m\}$,

$$(-1)^{a+j} \lambda(x_a) \lambda(x_j) > 0 \tag{4.3.6}$$

for all $j \in \{a, ..., a+q\}$.

(ii) $\dim P_n(\tilde{K}, \tilde{Z}) \leqslant 1$.

Proof. Suppose that (g, λ) is a non-zero element of $P_n(\tilde{K}, \tilde{Z})$. Then, by Theorem 4.2.3, $w_n(g, \lambda)$ has no zero interval. By Theorem 2.2.2(ii),

$$\tilde{Z}_n(w_n(g, \lambda)) \leqslant \tilde{S}_n(\lambda) \leqslant |\tilde{K} \cap [0, 2\pi)| = m \leqslant q+1.$$

However, q is even and $\tilde{S}^-(\lambda)$ is even. Therefore

$$\begin{aligned} q \geqslant \tilde{Z}_n(w_n(g, \lambda)) &= Z_n(w_n(g, \lambda), [0, 2\pi)) \\ &\geqslant |w_n(g, \lambda)^{-1}(0) \cap [0, 2\pi)| \geqslant |\tilde{Z} \cap [0, 2\pi)| = q. \end{aligned}$$

The equality (4.3.4) follows. Also, $Z_n(w_n(g, \lambda), z) = 1$ for each $z \in \tilde{Z}$ so that $w_n(g, \lambda)$ changes signs at each of its zeros (by 2.2.1) and so the condition (4.3.5) is satisfied. It also follows that $\tilde{S}^-(\lambda) = q$, so that at least q members of the set $\lambda(x_1), ..., \lambda(x_m)$ are non-zero. If $m = q$ then the inequalities (4.3.6) hold for any choice of $a \in \{1, ..., m\}$. If $m = q+1$ then $\lambda(x_{a-1}) \lambda(x_a) \geqslant 0$ for a unique $a \in \{1, ..., m\}$ which then satisfies condition (4.3.6). This proves (i).

If $(g, \lambda) \in \mathbb{R} \times \mathscr{M}_0$ is non-zero then $w_n(g, \lambda) \neq 0$. It follows that if $\dim P_n(\tilde{K}, \tilde{Z}) > 1$ then there exists a non-zero $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$ such that $w_n(g, \lambda)(y) = 0$ for at least one $y \notin \tilde{Z}$. This would contradict (4.3.4) of (i) and so (ii) is proved.

4.3.7. Lemma. *Suppose* $m = q$ *and*

$$\mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix} = 0. \tag{4.3.7}$$

If $r \in \{1, ..., q\}$ *then*

$$\mathscr{D}_n \begin{pmatrix} 1; & x_1 & \cdots & \cdots & \cdots & \cdots & \cdots & x_m \\ 1; & z_1 & \cdots & z_{r-1} & y & z_{r+1} & \cdots & z_q \end{pmatrix} = w_n(g_r, \lambda_r)(y) \tag{4.3.8}$$

for all $y \in \mathbb{R}$, *where* $(g_r, \lambda_r) \in P_n(\tilde{K}, \tilde{Z})$ *and*

$$(-1)^{r+j+1} \lambda_r(x_j) > 0 \tag{4.3.9}$$

for all $j \in \mathbb{Z}$.

Proof. That the equation (4.3.8) holds for some $(g_r, \lambda_r) \in P_n(\tilde{K}, \tilde{Z})$ follows from Proposition 4.3.3 and the assumption (4.3.7).

Let $J \in \{1, ..., m\}$ be such that, for some $k \in \mathbb{Z}$, $z_r \in (x_{J+km-1}, x_{J+km+1})$ (either there is one or there are two such J). Then

$$\lambda_r(x_J) = (-1)^{J+r} \mathscr{D}_n \begin{pmatrix} 1; & x_1 & \cdots & x_{J-1} & x_{J+1} & \cdots & x_m \\ 1; & z_1 & \cdots & z_{r-1} & z_{r+1} & \cdots & z_q \end{pmatrix}.$$

Now if $\tilde{K}' = \tilde{K} \backslash \{x_j : j \in (J + m\mathbb{Z})\}$ and $\tilde{Z}' = \tilde{Z} \backslash \{z_i : i \in (r + q\mathbb{Z})\}$ then $\tilde{K}'$ and $\tilde{Z}'$ also satisfy the condition (4.3.3) and so, by Theorem 4.3.2, $(-1)^{J+r+1} \lambda_r(x_J) > 0$. Therefore $\lambda_r \neq 0$ and, by Lemma 4.3.6, the condition (4.3.9) is satisfied.

5. THE CHARACTERISATION THEOREM

5.1. *Best Approximations in Case I*

The Cases I and II were defined in Section 4.1.

5.1.1. THEOREM. *Let $n > 1$. Suppose $v_0 \in \tilde{C} \backslash \tilde{W}_n$, and $u_0 \in \tilde{W}_n$. Then the following two conditions are equivalent.*

I(i) *u_0 is a best approximation to v_0 from $\tilde{W}_n$ and there exists $(g, \lambda) \in S(v_0, \tilde{W}_n)$ such that* $\operatorname{supp} \lambda \cap [0, 2\pi)$ *is finite and $w_n(g, \lambda)$ has a zero interval.*

I(ii) *There exist an interval $[a, b]$, where $a < b < a + 2\pi$, an integer $q > 1$, a sign $\varepsilon_M \in \{-1, 1\}$, points*

$$a = z_1 < \cdots < z_q = b$$

such that

$$u_0^{(n)}(x) = (-1)^{q+i} \varepsilon_M \qquad \textit{for almost all} \quad x \in (z_{i=1}, z_i)$$
$$\textit{and} \qquad i \in \{2, ..., q\}, \tag{5.1.1}$$

and points

$$a = x_1 < \cdots < x_m = b,$$

where $m = q + n - 1$, such that

$$(v_0 - u_0)(x_j) = (-1)^{m+j} \varepsilon_M \|v_0 - u_0\| \qquad \textit{for all} \quad j \in \{1, ..., m\}. \tag{5.1.2}$$

Proof. First assume that condition I(i) is satisfied by some (g, λ) and choose such a (g, λ) with $\operatorname{supp} \lambda$ as small as possible.

Let $[a, b]$ be a maximal interval such that $w_n(g, \lambda)^{-1}(0) \cap [a, b]$ is finite. So a and b are right and left end points respectively of zero intervals of $w_n(g, \lambda)$. Let $m = S^-(\lambda, [a, b]) + 1$. Let $q-2$ be the number of points of (a, b) at which $w_n(g, \lambda)$ changes sign. By Corollary 2.2.4,

$$q-2 \leqslant |w_n(g, \lambda)^{-1}(0) \cap (a, b)| \leqslant Z_n(w_n(g, \lambda), (a, b)) \\ \leqslant S^-(\lambda, [a, b]) - n = m - n - 1. \qquad (5.1.3)$$

That is, $m \geqslant q + n - 1$.

Now it is possible to choose points

$$a = x_1 < \cdots < x_m = b$$

and $\varepsilon_M = \operatorname{sgn} \lambda(b)$ such that $\{x_1, ..., x_m\} \subseteq \operatorname{supp} \lambda$ and

$$(-1)^{m+j} \varepsilon_M \lambda(x_j) > 0 \qquad \text{for} \quad j = 1, ..., m.$$

It follows from condition II(iii) of Theorem 3.1.1 that the equations (5.1.2) are satisfied.

Now consider the points

$$a = z_1 < \cdots < z_q = b,$$

where $z_2, ..., z_{q-1}$ are the points of (a, b) at which $w_n(g, \lambda)$ changes sign.

It remains to prove that $m = q + n - 1$ for this will imply that the inequalities of (5.1.3) are all equalities so that

$$w_n(g, \lambda)^{-1}(0) \cap (a, b) = \{z_2, ..., z_{q-1}\}.$$

Furthermore, by Lemma 4.2.1, if $\delta > 0$ is small then

$$\operatorname{sgn} w_n(g, \lambda)(x_m - \delta) = \operatorname{sgn} \lambda(x_m) = \varepsilon_M$$

and the equations (5.1.1) will follow from condition II(ii) of Theorem 3.1.1.

Let $\tilde{K}$ and $\tilde{Z}$ be the periodic extensions of the sets $\{x_1, ..., x_m\}$ and $\{z_1, ..., z_q\}$ respectively, as in paragraph 4.1.1. Suppose that $m > q + n - 1$. Then

$$|\tilde{Z} \cap (x_1, ..., x_m)| = q - 2 < m - 1 - n.$$

Then, by Lemma 4.2.4, there exist j, k such that $1 \leqslant j < k \leqslant m$ and the hypotheses of Theorem 4.2.2 are satisfied. Let $(g', \lambda') \in P_n(\tilde{K}, \tilde{Z})$ satisfy conditions (4.2.3), (4.2.4), and (4.2.5) of Theorem 4.2.2. Then $w_n(g', \lambda')$

changes sign in (x_j, x_k) at the points of the set $\tilde{Z} \cap (x_j, x_k)$ as does $w_n(g, \lambda)$. Replacing (g', λ') by $(-g', -\lambda')$ if necessary, we may assume that

$$w_n(g, \lambda)(y)\, w_n(g', \lambda')(y) \geqslant 0$$

for all $y \in (x_j, x_k)$. There are now two cases to consider: either (a) the signs of the two sequences $\lambda(x_j), ..., \lambda(x_k)$ and $\lambda'(x_j), ..., \lambda'(x_k)$ coincide or (b) they do not. In case (a) it follows from Proposition 3.1.3 that $(g', \lambda') \in S(v_0, \tilde{W}_n)$. But

$$\operatorname{supp} \lambda' \cap [x_1, x_1 + 2\pi) = \{x_j, ..., x_k\} \subseteq \operatorname{supp} \lambda \cap [x_1, x_1 + 2\pi).$$

By the minimality condition satisfied by supp λ it follows that $j = 1$ and $k = m$. However by Theorem 4.2.2

$$|\tilde{Z} \cap (x_j, x_k)| = k - j - n = m - 1 - n$$

so that $q = m + 1 - n$ which is a contradiction. In case (b) we can choose $\theta > 0$ to be the smallest positive number such that $\operatorname{supp}(\lambda + \theta\lambda') \neq \operatorname{supp} \lambda$. Then it follows that $(g + \theta g', \lambda + \theta\lambda') \in S(v_0, \tilde{W}_n)$ and that $w_n(g + \theta g', \lambda + \theta\lambda')$ shares a zero interval with $w_n(g, \lambda)$. This contradicts the minimality condition satisfied by supp λ. The proof that $m = q + n - 1$ is complete, as is the proof that condition I(i) implies condition I(ii). (The argument of this paragraph will recur in Section 5.2.)

Now assume that condition I(ii) is satisfied by $v_0, u_0, a, b, q, \varepsilon_M, m$, and the sets $\{x_1, ..., x_m\}$ and $\{z_1, ..., z_q\}$. It will be shown that there exists $(g', \lambda') \in \mathbb{R} \times \tilde{\mathcal{M}}_0$ which satisfies conditions II(i–iii) of Theorem 3.1.1. It will then follow that u_0 is a best approximation to v_0 from $\tilde{W}_n$.

Let $\tilde{K}$ and $\tilde{Z}$ be the periodic extensions, as in paragraph 4.1.1, of the sets $\{x_1, ..., x_m\}$ and $\{z_1, ..., z_q\}$ respectively; let h be as in paragraph 4.1.1. Then $h(m) = q$ and condition (C2) of Lemma 4.2.4, with $\varepsilon_F = \varepsilon_M$, is satisfied by the interval of integers $(1, m)$. Let j, k be the closest pair of integers in the interval $[1, m]$ such that condition (C) is satisfied. Then (j, k) satisfies the conclusions of Lemma 4.2.4; that is, j, k satisfy the hypotheses of Theorem 4.2.2. Therefore there exists $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$ such that conditions (4.2.3), (4.2.4), and (4.2.5) are satisfied. The pair j, k satisfies condition (C2) and so $(-1)^{q + h(k)} = (-1)^{m+k}$. It follows from this equation, from Eq. (5.1.1), and Eq. (4.2.4) that, for almost all $y \in (z_{i-1}, z_i) \cap (x_j, x_k)$ and each $i \in \mathbb{Z}$,

$$u_0^{(n)}(y) = (-1)^{m+k}\, \varepsilon_M \operatorname{sgn} \lambda(x_k) \operatorname{sgn} w_n(g, \lambda).$$

It follows from Eqs. (5.1.2) and (4.2.4) that, for $j' = j, ..., k$,

$$(v_0 - u_0)(x_{j'}) = (-1)^{m+k}\, \varepsilon_M \operatorname{sgn} \lambda(x_k) \operatorname{sgn} \lambda(x_{j'}) \|v_0 - u_0\|.$$

This shows that

$$(g', \lambda') = ((-1)^{m+k}\, \varepsilon_M \operatorname{sgn} \lambda(x_k) g,\ (-1)^{m+k}\, \varepsilon_M \operatorname{sgn} \lambda(x_k) \lambda)$$

satisfies I(i–iii) of Theorem 3.1.1. The proof of the theorem is complete.

From the proof of the theorem we obtain the following uniqueness result, which is incorporated in the main characterization Theorem 5.2.1.

5.1.2. COROLLARY. *If j, k is a closest pair of integers satisfying condition (C), with $\varepsilon_F = \varepsilon_M$, and u is also a best approximation to v_0 from $\tilde{W}_n$ then u coincides with u_0 on $[x_j, x_k]$.*

Proof. If (g', λ') is the element of $S(v_0, \tilde{W}_n)$ constructed in the proof of the theorem then $w_n(g', \lambda')$ has a finite number of zeros in (x_j, x_k). Therefore, by II(ii) of Theorem 3.1.1, $u^{(n)}$ and $u_0^{(n)}$ coincide on (x_j, x_k) so that the restriction of $u - u_0$ to (x_j, x_k) is a polynomial of degree $\leqslant n-1$. But, by II(iii) of Theorem 3.1.1, u and u_0 coincide on the set $\operatorname{supp} \lambda' \cap [x_j, x_k]$ which contains $k - j + 1$ points. However, by (4.2.7), $k - j + 1 \geqslant n + 1$. It follows that u and u_0 coincide on (x_j, x_k).

5.2. *The Characterization*

5.2.1. THEOREM. *Let $n > 1$. Let $v_0 \in \tilde{C} \setminus \tilde{W}_n$ and $u_0 \in \tilde{W}_n$. Then u_0 is a best approximation to v_0 from $\tilde{W}_n$ if and only if at least one of the following two conditions is satisfied*:

I. *There exist an interval $[a, b]$, where $a < b < a + 2\pi$; an integer $q > 1$; a sign $\varepsilon_M \in \{-1, 1\}$; points*

$$a = z_1 < \cdots < z_q = b$$

such that

$$u_0^{(n)}(x) = (-1)^{q+i} \varepsilon_M \qquad \text{for almost all} \quad x \in (z_{i-1}, z_i) \qquad \text{and} \qquad i \in \{2, ..., q\}; \tag{5.2.1}$$

and points

$$a = x_1 < \cdots < x_m = b,$$

where $m = q + n - 1$, such that

$$(v_0 - u_0)(x_j) = (-1)^{m+j}\, \varepsilon_M \, \|v_0 - u_0\| \qquad \text{for all} \quad j \in \{1, ..., m\}. \tag{5.2.2}$$

II. *There exist $m \in \mathbb{N}$; a sign $\varepsilon_M \in \{-1, 1\}$; a set $\tilde{K} = \{x_j; j \in \mathbb{Z}\}$ such that*

$$x_j < x_{j+1} \qquad \text{and} \qquad x_{j+m} = x_j + 2\pi \qquad \text{for all} \quad j \in \mathbb{Z}$$

and

$$(v_0 - u_0)(x_j) = (-1)^{m+j}\,\varepsilon_M\,\|v_0 - u_0\| \qquad \textit{for} \quad j = 1, ..., m; \tag{5.2.3}$$

an even integer q; *a sign* $\varepsilon_F \in \{-1, 1\}$; *and a set* $\tilde{Z} = \{z_i\colon i \in \mathbb{Z}\}$ *such that*

$$\begin{aligned} &z_i < z_{i+1} \qquad \textit{and} \qquad z_{i+q} = z_i + 2\pi \qquad \textit{for all} \quad i \in \mathbb{Z}; \\ &u^{(n)}(x) = (-1)^i\,\varepsilon_F \qquad \textit{for almost all} \quad x \in (z_{i-1}, z_i) \qquad \text{and all} \quad i \in \mathbb{Z}; \end{aligned} \tag{5.2.4}$$

and

$$|\tilde{Z} \cap (x_j, x_k)| > k - j - n \qquad \textit{whenever} \quad j < k < j + m; \tag{5.2.5}$$

and either $m = q + 1$,

$$\mathscr{D}_n \begin{pmatrix} 1; & x_2 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix} \mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{m-1} \\ 1; & z_1 \cdots z_q \end{pmatrix} > 0, \tag{5.2.6}$$

and

$$\varepsilon_M \varepsilon_F \mathscr{D}_n \begin{pmatrix} 1; & x_2 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix} > 0; \tag{5.2.7}$$

or $m = q$,

$$\mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix} = 0, \tag{5.2.8}$$

and, for some $r \in \{1, ..., q\}$ *and some* $y \in (z_{q-1}, z_q)$,

$$(-1)^{r+1}\,\varepsilon_M \varepsilon_F \mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{r-1} & x_r & x_{r+1} \cdots x_m \\ 1; & z_1 \cdots z_{r-1} & y & z_{r+1} \cdots z_q \end{pmatrix} > 0. \tag{5.2.9}$$

If condition I *is satisfied then it is satisfied by elements which also satisfy the additional condition*

$$x_j < z_j < x_{j-1+n} \qquad \text{for} \quad j = 2, ..., m - n,$$

and in this case each best approximation to v_0 *from* $\tilde{W}_n$ *coincides with* u_0 *on the interval* $[a, b]$.

If condition II *is satisfied then* u_0 *is the unique best approximation to* v_0 *from* $\tilde{W}_n$.

Proof. Suppose that u_0 is a best approximation to v_0 from $\tilde{W}_n$ but that condition I is not satisfied. It must be shown that condition II is satisfied. By Theorem 5.1.1 there exists $(g, \lambda) \in S(v_0, \tilde{W}_n)$ such that supp λ is minimal, and so supp $\lambda \cap [0, 2\pi)$ is finite by Theorem 3.1.7, and the associated function $w_n(g, \lambda)$ has maximal zero set in the sense of Theorem 3.1.5. Then $w_n(g, \lambda)$ has no zero interval.

Now let $m \in \mathbb{N}$ and supp $\lambda = \tilde{K} = \{x_j : j \in \mathbb{Z}\}$ be such that

$$x_j < x_{j+1} \qquad \text{and} \qquad x_{j+m} = x_j + 2\pi \qquad \text{for all} \quad j \in \mathbb{Z}.$$

Furthermore, if the signs of $\lambda(x_j)$, $j \in \mathbb{Z}$, do not alternate we can choose the indexing of $\tilde{K}$ so that $\lambda(x_1)\,\lambda(x_m) > 0$. Let $\varepsilon_M = \operatorname{sgn} \lambda(x_m)$. Let $\tilde{Z}$ be the set of points at which the periodic spline function $w_n(g, \lambda)$ changes sign. Then, for some even integer q, $\tilde{Z} = \{z_i ; i \in \mathbb{Z}\}$ where

$$z_i < z_{i+1} \qquad \text{and} \qquad z_{i+q} = z_i + 2\pi \qquad \text{for all} \quad i \in \mathbb{Z}.$$

Thus $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$. Let $\varepsilon_F \in \{-1, 1\}$ be such that

$$(-1)^{q+i}\, \varepsilon_F w_n(g, \lambda)(y) \geqslant 0 \qquad \text{for all} \quad y \in (z_{i-1}, z_i) \qquad \text{and all} \quad i \in \mathbb{Z}.$$

The crucial steps in the proof of the theorem are now provided by the following lemma.

5.2.2. Lemma. (i) *$|\tilde{Z} \cap (x_j, x_k)| > k - j - n$ whenever $j < k < j + m$.*

(ii) *Either $m = q$ or $m = q + 1$.*

Proof of (i). Assume that (i) is false. Let j, k be the closest pair of integers such that $|\tilde{Z} \cap (x_j, x_k)| \leqslant k - j - n$. Then by Theorem 4.2.2 there exists $(g', \lambda') \in P_n(\tilde{K}, \tilde{Z})$ such that conditions (4.2.3), (4.2.4), and (4.2.5) are satisfied. Then $w_n(g', \lambda')$ changes sign at the points of $\tilde{Z} \cap (x_j, x_k)$, as does $w_n(g, \lambda)$. We may suppose, replacing (g', λ') by $(-g', -\lambda')$ if necessary, that

$$w_n(g, \lambda)(y)\, w_n(g', \lambda')(y) \geqslant 0 \qquad \text{for all} \quad y,$$

so that

$$(-1)^{q+i}\, \varepsilon_F w_n(g', \lambda')(y) > 0$$

$$\text{for all} \quad y \in (z_{i-1}, z_i) \cap (x_j, x_k) \qquad \text{and all} \quad i \in \mathbb{Z}.$$

Now supp $\lambda' \subseteq \tilde{K} = \operatorname{supp} \lambda$. If

$$\operatorname{supp} \lambda'^{+} \subseteq \operatorname{supp} \lambda^{+}, \qquad \operatorname{supp} \lambda'^{-} \subseteq \operatorname{supp} \lambda^{-},$$

then it follows by Proposition 3.1.3 that $(g', \lambda') \in S(v_0, \tilde{W}_n)$, which contradicts the fact that condition I (and so also condition I(i) of Theorem 5.1.1) is not satisfied, so that the associated function $w_n(g', \lambda')$ cannot have a zero interval. Therefore either

$$\operatorname{supp} \lambda'^{+} \nsubseteq \operatorname{supp} \lambda^{+} \qquad \text{or} \quad \operatorname{supp} \lambda'^{-} \nsubseteq \operatorname{supp} \lambda^{-}.$$

Now choose $\theta > 0$ to be the smallest number such that either

$$\operatorname{supp}(\lambda + \theta\lambda')^{+} \neq \operatorname{supp} \lambda^{+}$$

or

$$\operatorname{supp}(\lambda + \theta\lambda')^{+} \neq \operatorname{supp} \lambda^{-}.$$

Then

$$\operatorname{supp}(\lambda + \theta\lambda')^{+} \subseteq \operatorname{supp} \lambda^{+}$$

and

$$\operatorname{supp}(\lambda + \theta\lambda')^{-} \subseteq \operatorname{supp} \lambda^{-}.$$

It follows, again from Proposition 3.1.3, that $(g + \theta g', \lambda + \theta\lambda') \in S(v_0, \tilde{W}_n)$, which contradicts the minimality of supp λ. This completes the proof of (i).

Proof of (ii). By the definitions of $\tilde{K}$, m, $\tilde{Z}$, and q, and by Theorem 2.2.2(ii),

$$q \leqslant \tilde{Z}_n(w_n(g, \lambda)) \leqslant \tilde{S}^{-}(\lambda) \leqslant m.$$

Now it will be proved by contradiction that $m \leqslant q + 1$. Suppose, on the contrary, that $m > q + 1$. Let $\tilde{K}^*$ be the periodic set obtained by deleting from $\tilde{K}$ the points $x_{q+2}, ..., x_m$ and all their translates by multiples of 2π. Then we can write $\tilde{K}^* = \{x_j^* : j \in \mathbb{Z}\}$ where $x_j^* = x_j$ for $j = 1, ..., q + 1$ and $x_{j+q+1}^* = x_j^* + 2\pi$ for all $j \in \mathbb{Z}$. If $j \in \mathbb{Z}$ then $x_j^* = x_{J(j)}$ where $J(j) = j + [j/q+1](m - q - 1)$. Thus if $j < k < j + q + 1$ then $x_j^* < x_k^* < x_k^* + 2\pi$ and $k - j \leqslant J(k) - J(j) < m$ so that, by (i) of the lemma,

$$|\tilde{Z} \cap (x_j^*, x_k^*)| = |\tilde{Z} \cap (x_{J(j)}, x_{J(k)})| > J(k) - J(j) - n \geqslant k - j - n.$$

By Proposition 4.3.3 and Lemma 4.3.5 there exists a (g', λ') of $S_n(\tilde{K}^*, \tilde{Z})$ such that $\lambda' \neq 0$. It also follows from Lemma 4.3.5 and Theorem 4.3.2 that $w_n(g', \lambda')^{-1}(0) = \tilde{Z}$ and that $w_n(g', \lambda')$ changes sign at each of its zeros. So we may assume, replacing (g', λ') by $(-g', -\lambda')$ if necessary, that

$$w_n(g, \lambda)(y)\, w_n(g', \lambda')(y) > 0$$

for all $y \notin \tilde{Z}$. The argument now runs as in the proofs of Theorem 5.1.1 and part (i) of the lemma, giving a contradiction to the fact that supp λ is minimal. The proof of part (ii) of the lemma is complete.

The proof that condition II is satisfied can now be completed.

Suppose that $m = q+1$. Let (g', λ') be the pair such that

$$w_n(g', \lambda') = \mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{m-1} & x_m \\ 1; & z_1 \cdots z_q & y \end{pmatrix},$$

so that $(g', \lambda') \in P_n(\tilde{K}, \tilde{Z})$ and $\lambda' \neq 0$ (by Proposition 4.3.3 and Lemma 4.3.5). By Lemma 4.3.6 $\dim P_n(\tilde{K}, \tilde{Z}) = 1$ so $(g, \lambda) = (\alpha g', \alpha \lambda')$ and $w_n(g, \lambda) = \alpha w_n(g', \lambda')$ for some $\alpha \in \mathbb{R} \backslash \{0\}$. It follows from the definitions of ε_F and ε_M that

$$\lambda(x_1)\, \lambda(x_m) > 0$$

and

$$\varepsilon_M \lambda(x_1)\, \varepsilon_F w_n(g, \lambda)(y) \leqslant 0$$

for all $y \in (z_q, z_{q+1})$. In these inequalities g, λ can be replaced by g', λ', and the conditions (5.2.6) and (5.2.7) follow from Proposition 4.3.3 and Theorem 4.3.2.

Now suppose that $m = q$. The pair (g, λ) is a non-zero element of $P_n(\tilde{K}, \tilde{Z})$ and so $\dim P_n(\tilde{K}, \tilde{Z}) \geqslant 1$. By (i) of Lemma 5.2.2 and (ii) of Lemma 4.3.4 $\dim P_n(\tilde{K}, \tilde{Z}) = 1$. If r is any of 1, ..., q let $(g_r, \lambda_r) \in P_n(\tilde{K}, \tilde{Z})$ be as in Lemma 4.3.7. Then $(g, \lambda) = (\alpha g_r, \alpha \lambda_r)$ for some $\alpha \in \mathbb{R} \backslash \{0\}$ and it follows from the definitions of ε_F and ε_M and from Lemma 4.3.7 and Proposition 4.3.3 that (5.2.9) is satisfied by every $y \in (z_{q-1}, z_q)$.

The proof that if u_0 is a best approximation to v_0 from $\tilde{W}_n$ then either Condition I or II is satisfied is complete. If I is satisfied then by Theorem 5.1.1 u_0 is a best approximation. Suppose that Condition II is satisfied. By (5.2.5) the assumptions of 4.3.4 are satisfied by $\tilde{K}$ and $\tilde{Z}$.

Suppose that Condition II is satisfied with $m = q+1$. Let

$$\alpha = \varepsilon_M \mathscr{D}_n \begin{pmatrix} 1; & x_2 \cdots x_m \\ 1; & z_1 \cdots z_q \end{pmatrix}$$

and let $(g, \lambda) \in P_n(\tilde{K}, \tilde{Z})$ be the pair (non-zero by (5.2.5) and Lemma 4.3.5) such that

$$w_n(g, \lambda) = \alpha \mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{m-1} & x_m \\ 1; & z_1 \cdots z_q & y \end{pmatrix}.$$

Then by (5.2.6), using Proposition 4.3.3, $\lambda(x_1)\,\lambda(x_m)>0$. It follows that (4.3.6) of Lemma 4.3.6 is satisfied by $a=1$ so that

$$(-1)^{m+j}\,\varepsilon_M\lambda(x_j)>0 \qquad \text{for} \quad j=1, ..., m.$$

It now follows from (5.2.3) that λ satisfies Condition II(iii) of Theorem 3.1.1. Now (5.2.8) means that $\alpha\varepsilon_F>0$. It now follows from (5.2.3), using Theorem 4.3.2(i), that (g, λ) satisfies Condition II(ii) of Theorem 3.1.1. So, by Theorem 3.1.1, u_0 is a best approximation.

Finally, suppose that Condition II is satisfied with $m=q$. Let $r\in\{1, ..., q\}$ be such that (5.2.9) holds for some $y\in(z_{q-1}, z_q)$. Condition (5.2.5) allows us to appeal to Lemma 4.3.7. Let $\alpha=(-1)^{r+1}\varepsilon_M$ and let $(g, \lambda)=(\alpha g_r, \alpha\lambda_r)$ where (g_r, λ_r) is as in Lemma 4.3.7. Then $w_n(g, \lambda)=\alpha w_n(g_r, \lambda_r)$ and

$$\varepsilon_M(-1)^j\lambda(x_j)>0 \qquad \text{for all} \quad j\in\mathbb{Z}.$$

It follows from (5.2.3) that λ satisfies II(iii) of Theorem 3.1.1. By (5.2.9) and (4.3.5) of Lemma 4.3.6

$$(-1)^i\,\varepsilon_F w_n(g, \lambda)(y)>0 \qquad \text{for all} \quad y\in(z_{i-1}, z_i) \qquad \text{and all} \quad i\in\mathbb{Z}.$$

So, by (5.2.4), (g, λ) satisfies Condition II(ii) of Theorem 3.1.1. Therefore u_0 is a best approximation.

The uniqueness statements of the theorem are given in Case I by Corollary 5.1.2 and Lemma 4.2.5, and in Case II by Corollary 3.1.2. The proof of the theorem is complete.

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APPENDIX

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This appendix is devoted to a more geometrical characterization of best uniform approximations to periodic continuous functions by functions from the class $\widetilde{W}_n$, which, as we show, is in fact equivalent to the characterization of Theorem 5.2.1 above.

A.1. THEOREM [8]. *Let* $v_0 \in \tilde{C} \backslash \tilde{W}_n, n \geqslant 2$. *A function* $u_0 \in \tilde{W}_n$ *is a best uniform approximation to* v_0 *from* $\tilde{W}_n$ *if and only if at least one of the following is true.*

A. *There exists an interval* $[a, b], 0 < b - a < 2\pi$ *such that*

$$\|v_0 - u_0\|_{\tilde{C}} = \|v_0 - u_0\|_{C[a, b]}$$

and $u_0|_{[a, b]}$ *is a best approximation to* $v_0|_{[a, b]}$ *from* $W_n([a, b])$.

B. *There exists* $v \in \mathbb{N}$ *such that*

(i) u_0 *is a periodic perfect spline of degree n with exactly* $2v$ *simple knots over a period,*

$$z_1 < \cdots < z_{2v} < z_1 + 2\pi (= z_{2v+1}),$$

such that

$$u_0^{(n)}(x) = (-1)^{i+1} \delta \qquad \text{a.e. in} \quad (z_i, z_{i+1}), \qquad i = 1, ..., 2v, \tag{A.1}$$

for some $\delta \in \{-1, 1\}$.

(ii) *There exist* $x_1 < \cdots < x_{2v+1} \leqslant x_1 + 2\pi$ *such that*

$$(v_0 - u_0)(x_j) = (-1)^j \varepsilon \|v_0 - u_0\|, \qquad j = 1, ..., 2v+1, \tag{A.2}$$

for some $\varepsilon \in \{-1, 1\}$.

(iii) *There exists a periodic spline* u_1 *of degree* $n-1$ *with simple knots* $\{z_i\}_{i=1}^{2v}$, *which has simple zeros at points* $x_1', x_2, x_3, ..., x_{2v}$ *for some* $x_1' \in [x_{2v+1} - 2\pi, x_1]$ *and does not vanish at any other point of the interval* $(x_1', x_1' + 2\pi)$.

(iv) *There exists a periodic spline* u_2 *of degree n with simple knots* $\{z_i\}_{i=1}^{2v}$ *which satisfies the conditions*

$$(-1)^{i+1} \delta u_2^{(n)}(x) > 0 \qquad \text{a.e. in} \quad (z_i, z_{i+1}), i = 1, ..., 2v, \tag{A.3}$$

and

$$\varepsilon u_2(x_1') < 0, \quad \varepsilon u_2(x_2) > 0, \quad \varepsilon u_2(x_3) < 0, \quad ..., \quad \varepsilon u_2(x_{2v}) > 0. \tag{A.4}$$

Furthermore, if condition B *is satisfied, then* u_0 *is the unique best approximation to* v_0 *from* $\tilde{W}_n$.

There exists an unpublished proof of A.1 which largely follows the lines of Satte's proof for the non-periodic case and is based on a characterization of best uniform approximation by periodic splines with fixed simple knots [7]. We do not present the original proof of A.1 here because it is lengthy. For some closely related results on approximation from classes of periodic functions defined by integral operators with strictly CVD kernels, see [5, 6].

In order to prove the equivalence of Theorem 5.2.1 and Theorem A.1, we need some definition and auxiliary results from [7]. Let $\tilde{Z}_{2\nu} = \{z_i : i \in \mathbb{Z}\}$ and $\tilde{T}_{2\nu} = \{t_j : j \in \mathbb{Z}\}$ be two periodic sequences such that

$$z_1 < \cdots < z_{2\nu} < z_1 + 2\pi \quad \text{and} \quad z_{i+2\nu} = z_i + 2\pi \quad \text{for all} \quad i \in \mathbb{Z},$$
$$t_1 < \cdots < t_{2\nu} < t_1 + 2\pi \quad \text{and} \quad z_{j+2\nu} = t_j + 2\pi \quad \text{for all} \quad j \in \mathbb{Z}.$$

Denote by $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$, the set of periodic spline functions of degree $n-1$ with simple knots at the points of the set $\tilde{Z}_{2\nu}$. Then $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ is a finite-dimensional linear space, dim $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu}) = 2\nu$. The sequence $\tilde{T}_{2\nu}$ is called an I-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ if for any given real numbers $a_1, ..., a_{2\nu}$ there exists a unique spline $u \in \mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ such that $u(t_j) = a_j$ for $j = 1, ..., 2\nu$. The sequence $\tilde{T}_{2\nu}$ is said to be an NI-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ if there exists a spline function $w \in \mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ (we call it an *undulating spline*) which has simple zeros exactly at the points of the set $\tilde{T}_{2\nu}$. It follows from Proposition 3 and the Remark after Lemma 4 in [7] that for any given NI-set an undulating spline is determined uniquely up to a nonzero real factor. Note that although a sequence $\tilde{T}_{2\nu}$ evidently cannot be an I-set and an NI-set simultaneously, there exist such sequences $\tilde{T}_{2\nu}$ that are neither I-sets nor NI-sets. The latter happens in the case that a non-zero spline with zero intervals vanishes at the points $t_1, ..., t_{2\nu}$.

A.2. Lemma. [7], Lemma 4]. *If the sequence $\tilde{T}_{2\nu}$ satisfies the condition*

$$|[z_i, z_{i+\mu}] \cap \tilde{T}_{2\nu}| \leqslant n + \mu - 1, \qquad i, \mu = 1, ..., 2\nu, \tag{A.5}$$

then $\tilde{T}_{2\nu}$ is either an I-set or an NI-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$.

A.3. Lemma [7, Remark after Lemma 4]. *If $\tilde{T}_{2\nu}$ is an NI-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$, then*

$$|[z_i, z_{i+\mu}] \cap \tilde{T}_{2\nu}| \leqslant n + \mu - 2, \qquad i, \mu = 1, ..., 2\nu, \tag{A.6}$$

The following lemma is a strengthened version of Lemma 3 in [7].

A.4. Lemma. *Let $\tilde{T}_{2\nu} = \{t_j : j \in \mathbb{Z}\}$ be an NI-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ and let $\tilde{T}'_{2\nu} = \{t'_j : j \in \mathbb{Z}\}$ be a periodic sequence, $t'_{j+2\nu} = t'_j + 2\pi$ for all $j \in \mathbb{Z}$. If*

$$t_j \leqslant t'_j \leqslant t_{j+1}, \qquad j = 1, ..., 2\nu,$$

with $\tilde{T}'_{2\nu} \neq \tilde{T}_{2\nu}$, then $\tilde{T}'_{2\nu}$ is an I-set.

Proof. In view of Lemma A.3, $\tilde{T}'_{2\nu}$ satisfies (A.5). Therefore, by Lemma A.2, it suffices to show that $\tilde{T}'_{2\nu}$ cannot be an *NI*-set. Suppose the contrary. Let w and w_1 be undulating splines which correspond to $\tilde{T}_{2\nu}$ and $\tilde{T}'_{2\nu}$ respectively. As in the proof of Lemma 3 in [7], find a linear combination $\tilde{w} = \alpha w + \beta w_1$ such that $\tilde{w}^{(n-1)}$ possesses at most $2\nu - 2$ cyclic sign changes. However, $\tilde{w}$ evidently has 2ν distinct zeros $t''_j \in [t_j, t_{j+1}]$, $j = 1, ..., 2\nu$, over a period. The periodic sequence $\tilde{T}''_{2\nu} = \{t''_j : j \in \mathbb{Z}\}$ generated by these zeros is, by Lemma A.2, either an *I*-set or an *NI*-set. If it is an I-set, then $\tilde{w} \equiv 0$, hence $\alpha w = -\beta w_1$, which contradicts the assumption that $\tilde{T}'_{2\nu} \neq \tilde{T}_{2\nu}$. If $\tilde{T}''_{2\nu}$ is an NI-set, then $\tilde{w}$ has 2ν cyclic sign changes and so does $\tilde{w}^{(n-1)}$, a contradiction. The proof is complete.

The next lemma follows immediately from the well-known representation of periodic splines by the Bernoulli polynomials D_n (see, for example, [1, Chapter 8]).

A.5. Lemma. *$\tilde{T}_{2\nu}$ is an I-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ if and only if*

$$\mathscr{D}_n \begin{pmatrix} 1; & t_1 \cdots t_{2\nu} \\ 1; & z_1 \cdots z_{2\nu} \end{pmatrix} \neq 0.$$

It follows from (4.3.1) that

$$\mathscr{D}_n \begin{pmatrix} 1; & t_1 \cdots t_{2\nu} \\ 1; & z_1 \cdots z_{2\nu} \end{pmatrix} = (-1)^n \, \mathscr{D}_n \begin{pmatrix} 1; & z_1 \cdots z_{2\nu} \\ 1; & t_1 \cdots t_{2\nu} \end{pmatrix}.$$

This fact, together with the simple observation that (A.6) implies

$$|[t_j, t_{j+\mu}] \cap \tilde{Z}_{2\nu}| \leqslant n + \mu - 2, \qquad j, \mu = 1, ..., 2\nu,$$

leads, in view of Lemmas A.2, A.3, and A.5, to a kind of duality presented in the following lemma.

A.6. Lemma. *$\tilde{T}_{2\nu}$ is an I-set (NI-set) with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ if and only if $\tilde{Z}_{2\nu}$ is an I-set (NI-set) with respect to $\mathscr{S}_{n-1}(\tilde{T}_{2\nu})$.*

A dual variant of Lemma A.4 follows immediately.

A.7. Lemma. *Let $\tilde{T}_{2\nu} = \{t_j : j \in \mathbb{Z}\}$ be an NI-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ and let $\tilde{Z}'_{2\nu} = \{z'_i : i \in \mathbb{Z}\}$ be a periodic sequence, $z'_{i+2\nu} = z'_i + 2\pi$ for all $i \in \mathbb{Z}$. If*

$$z_i \leqslant z'_i \leqslant z_{i+1}, \qquad i = 1, ..., 2\nu,$$

with $\tilde{Z}'_{2\nu} \neq \tilde{Z}_{2\nu}$, then $\tilde{T}_{2\nu}$ is an I-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}'_{2\nu})$.

Proof of the Equivalence of Theorems 5.2.1 *and* A.1. First, the equivalence of Condition I of Theorem 5.2.1 and Condition A of A.1 evidently follows from Sattes's characterization theorem.

Next, it is readily seen that q, ε_F, and $(-1)^m \varepsilon_M$ in Condition II of Theorem 5.2.1 correspond to $2v, \delta$, and ε in Condition B of A.1 respectively. Two cases $m=q+1$ and $m=q$ in Theorem 5.2.1 can be recognized as strict inequality $x_{2v+1}<x_1+2\pi$ and equality $x_{2v+1}=x_1+2\pi$ in Condition B(ii) of Theorem A.1. Thus, (5.2.3) and (5.2.4) can be identified with (A.2) and (A.1)

It is not hard to check that condition (5.2.5) can be restated as follows:

$$|[z_i, z_{i+\mu}] \cap \tilde{K}| \leqslant n+\mu-1, \qquad i, \mu=1, ..., q. \tag{A.7}$$

We consider two cases.

Case 1: $m=q$ $(x_{2v+1}=x_1+2\pi)$. We first show that (5.2.5) & (5.2.8) $\Leftrightarrow$ (iii). We have $\tilde{K}=\tilde{X}_{2v} \overset{\text{def}}{=} \{x_j : j \in \mathbb{Z}\}$, where $x_{j+2v}=x_j+2\pi$, $j \in \mathbb{Z}$. Since $x_1'=x_1$, (iii) means that $\tilde{X}_{2v}$ is an *NI*-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2v})$. If (iii) holds, then Lemma A.5 implies (5.2.8) and Lemma A.3 ensures (A.7), so that (5.2.5) is also true. Conversely, (iii) follows from (5.2.5) and (5.2.8) in view of Lemmas A.2 and A.5.

Now suppose (iii) to hold and prove the equivalence of (iv) and (5.2.9). By Lemmas A.7 and A.5,

$$\mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{2v} \\ 1; & z_1' \cdots z_{2v}' \end{pmatrix} \neq 0 \tag{A.8}$$

for any $\tilde{Z}_{2v}'$ satisfying

$$z_j \leqslant z_j' \leqslant z_{j+1}, \qquad j=1, ..., 2v, \qquad \text{and} \qquad \tilde{Z}_{2v}' \neq \tilde{Z}_{2v}, \tag{A.9}$$

and thus is of one fixed sign throughout this domain. It is easily seen that (5.2.9) determines this sign as $\varepsilon_M \varepsilon_F$. Therefore, (5.2.9) is equivalent to the following:

$$\varepsilon\delta\mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{2v} \\ 1; & z_1' \cdots z_{2v}' \end{pmatrix} > 0 \tag{A.10}$$

for any $\tilde{Z}_{2v}'$ satisfying (A.9).

On the other hand, any spline function $u \in \mathscr{S}_n(\tilde{Z}_{2v})$ can be uniquely represented in the form

$$u(x)=d_0+\sum_{i=1}^{2v} d_i(D_{n+1}(x-z_i)-D_{n+1}(x-z_{i+1})), \tag{A.11}$$

where

$$d_0 = \int_0^{2\pi} u(x)\,dx, \qquad d_i = u^{(n)}(x) \qquad \text{for any} \quad x \in (z_i, z_{i+1}), \qquad i = 1, ..., 2\nu,$$

with

$$\sum_{i=1}^{2\nu} d_i(z_{i+1} - z_i) = 0. \tag{A.12}$$

Let us consider the equalities

$$\begin{gathered} \sum_{i=1}^{2\nu} d_i(z_{i+1} - z_i) = 0, \\ d_0 + \sum_{i=1}^{2\nu} d_i(D_{n+1}(x_j - z_i) - D_{n+1}(x_j - z_{i+1})) = u(x_j), \qquad j = 1, ..., 2\nu, \end{gathered} \tag{A.13}$$

as a system of linear equations in the unknowns $d_0, d_1, ..., d_{2\nu}$. The determinant of this system,

$$\Delta = \begin{vmatrix} 0 & z_2 - z_1 & \cdots & z_1 + 2\pi - z_{2\nu} \\ 1 & D_{n+1}(x_1 - z_1) - D_{n+1}(x_1 - z_2) & \cdots & D_{n+1}(x_1 - z_{2\nu}) - D_{n+1}(x_1 - z_{2\nu+1}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & D_{n+1}(x_{2\nu} - z_1) - D_{n+1}(x_{2\nu} - z_2) & \cdots & D_{n+1}(x_{2\nu} - z_{2\nu}) - D_{n+1}(x_{2\nu} - z_{2\nu+1}) \end{vmatrix}$$

$$= \int_{z_1}^{z_2} \cdots \int_{z_{2\nu}}^{z_1 + 2\pi} \mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{2\nu} \\ 1; & z'_1 \cdots z'_{2\nu} \end{pmatrix} dz'_{2\nu} \cdots dz'_1,$$

is nonzero because of (A.8).

If (5.2.9) holds, then

$$\varepsilon\delta\Delta > 0.$$

Since $\Delta \neq 0$, there exists a spline function $u = u_2 \in \mathscr{S}_n(\tilde{Z}_{2\nu})$ satisfying A.4 (where $x'_1 = x_1$). Then u_2 has 2ν sign changes and the same is true for $u_2^{(n)}$. Because of this, $d_i \neq 0$, $i = 1, ..., 2\nu$. Solving for d_i in (A.13), we obtain

$$d_i = \Delta^{-1} \sum_{j=1}^{2\nu} (-1)^{i+j} u_2(x_j) \int_{z_1}^{z_2} \cdots \int_{z_{i-1}}^{z_i} \int_{z_{i+1}}^{z_{i+2}} \cdots \int_{z_{2\nu}}^{z_1 + 2\pi}$$

$$\mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{j-1} x_{j+1} \cdots x_{2\nu} \\ 1; & z'_1 \cdots z'_{i-1} z'_{i+1} \cdots z'_{2\nu} \end{pmatrix} dz'_{2\nu} \cdots dz'_{i+1} dz'_{i-1} \cdots dz'_1.$$

It follows from Theorem 4.3.2 above that the integrals in the right-hand side are all non-positive. Therefore,

$$(-1)^{i+1}\,\delta d_i > 0, \qquad i = 1, ..., 2\nu.$$

Since $d_i = u^{(n)}(x)$, $x \in (z_i, z_{i+1})$, (A.3) follows, and (iv) is obtained.

Conversely, if (iv) holds, then the same matrix computation shows that $\varepsilon\delta\Delta > 0$, i.e.,

$$\varepsilon\delta \int_{z_1}^{z_2} \cdots \int_{z_{2\nu}}^{z_1+2\pi} \mathscr{D}_n \begin{pmatrix} 1; & x_1 \cdots x_{2\nu} \\ 1; & z'_1 \cdots z'_{2\nu} \end{pmatrix} dz'_{2\nu} \cdots dz'_1 > 0,$$

which implies (A.10).

Case 2: $m = q+1$ $(x_{2\nu+1} < x_1 + 2\pi)$. The function

$$w(x) = \mathscr{D}_n \begin{pmatrix} 1; & x & x_2 \cdots x_{2\nu} \\ 1; & z_1 & z_2 \cdots z_{2\nu} \end{pmatrix}$$

is a spline in $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$ which vanishes at the points $x_2, x_2, ..., x_{2\nu}$. If (5.2.6) holds, then $w(x_{2\nu+1} - 2\pi)\, w(x_1) < 0$, hence there exists another zero x'_1 of $w(x)$ which is located inside $(x_{2\nu+1} - 2\pi, x_1)$. Set $\tilde{X}'_{2\nu} = \{x'_j : j \in \mathbb{Z}\}$, where x'_1 is that zero, $x'_j = x_j$, $j = 2, ..., 2\nu$, and $x'_{j+2\nu} = x'_j + 2\pi$ for all $j \in \mathbb{Z}$. If (5.2.5) also holds, then it follows from Lemmas A.2 and A.5 that $\tilde{X}'_{2\nu}$ is an *NI*-set with respect to $\mathscr{S}_{n-1}(\tilde{Z}_{2\nu})$, i.e., (iii) is valid. Conversely, if (iii) holds, then $w(x)$ is an undulating spline with zeros at the points of $\tilde{X}'_{2\nu}$, so that necessarily $w(x_{2\nu+1} - 2\pi)\, w(x_1) < 0$, which gives (5.2.6). The validity of (5.2.5) follows from (iii) in view of Lemma A.3. Thus, (5.2.5) & (5.2.6) $\Leftrightarrow$ (iii).

What is left is to check the equivalence of (5.2.7) and (iv) under the supposition that (iii) holds. To this end we can clearly apply the same argument as in Case 1, if it is shown that (5.2.7) is equivalent to the condition

$$\varepsilon\delta\mathscr{D}_n \begin{pmatrix} 1; & x'_1 & x_2 \cdots x_{2\nu} \\ 1; & z'_1 & z'_2 \cdots z'_{2\nu} \end{pmatrix} > 0$$

for any $\tilde{Z}'_{2\nu}$ satisfying

$$z_i \leqslant z'_i \leqslant z_{i+1}, \quad i = 1, ..., 2\nu, \qquad \text{and} \qquad \tilde{Z}'_{2\nu} \neq \tilde{Z}_{2\nu}.$$

The latter is readily seen from the next calculation, if it is considered that $\varepsilon_M \varepsilon_F = -\varepsilon\delta$ if m is odd.

$$\operatorname{sgn} \mathscr{D}_n \begin{pmatrix} 1; & x_2 \cdots x_{2\nu+1} \\ 1; & z_1 \cdots z_{2\nu} \end{pmatrix} = -\operatorname{sgn} \mathscr{D}_n \begin{pmatrix} 1; & x_{2\nu+1} - 2\pi & x_2 & \cdots & x_{2\nu} \\ 1; & z_1 & z_2 & \cdots & z_{2\nu} \end{pmatrix}$$

$$= -\operatorname{sgn} \mathscr{D}_n \begin{pmatrix} 1; & x_1' - \eta & x_2 - \eta & \cdots & x_{2\nu} - \eta \\ 1; & z_1 & z_2 & \cdots & z_{2\nu} \end{pmatrix}$$

$$= -\operatorname{sgn} \mathscr{D}_n \begin{pmatrix} 1; & x_1' & x_2 & \cdots & x_{2\nu} \\ 1; & z_1 + \eta & z_2 + \eta & \cdots & z_{2\nu} + \eta \end{pmatrix},$$

if $\eta > 0$ is small enough.

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